On Manifolds Of Nonpositive Curvature

by

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Abstract

In this thesis, we survey on some geometric properties of complete Riemannian manifolds of nonpositive curvature. One of the themes of the survey is to study first the simply-connected ones by considering their boundaries at infinity. The geometry of a complete simply-connected Riemannian manifold $\tilde{M}$ of nonpositive curvature can be reflected by the action of isometries and of geodesic symmetries on the boundary at infinity $\tilde{M}(\infty)$. Then, we can study a nonsimply-connected manifold M of nonpositive curvature by studying the action of isometries of the fundamental group of M on the universal cover $\tilde{M}$ and $\tilde{M}(\infty)$.

The first chapter is an introduction and describes the boundary at infinity as a natural extension of convexity methods which were introduced by R. Bishop and B. O'Neill. In the second chapter, we study symmetric spaces of noncompact type, which form an important class of manifolds of nonpositive curvature. The particular example of $\tilde{M}_n = SL(n,\mathbb{R})/SO(n,\mathbb{R})$ is described in detail to explain the various concepts and properties. In the third chapter, we report on the method of group action on the boundary at infinity, which yields, for example, results on geodesic flows and a characterization of symmetric spaces of noncompact type of rank at least 2. We end the thesis by giving an outline of the rigidity theorems of Mostow and Gromov.

Contents

Chapter 1

Introduction

1.1 Riemannian Manifolds

A Riemannian manifold is a differentiable manifold M with a scalar product $<,>$ on each tangent space T_pM, for all $p \in M$.

This Riemannian metric $<,>$ defines a distance between any two points p, q in M. Namely,

$$d(p,q) = \inf L(\gamma)$$

where $\gamma : [a,b] \longrightarrow M$ is a differentiable curve such that $\gamma(a) = p, \gamma(b) = q$ and

$$L(\gamma) = \int_a^b < \gamma'(t), \gamma'(t) >^{\frac{1}{2}} dt.$$

1.1.1 Completeness

The Riemannian manifold M is called complete if it is a complete metric space with respect to $d(,)$

Completeness of M can be charaterized in terms of the geodesics of M, in view of the following basic theorem.

Theorem 1.1.1 (Hopf-Rinow) *Let M be a Riemannian manifold, then the following conditions are equivalent:*

a, M is a complete metric space with respect to $d(,)$.

b, For some $p \in M$, exp_p is defined on all of T_pM.

c, For all $p \in M$, exp_p is defined on all of T_pM.

d, Every bounded subset of M is compact.

Further, any of these conditions implies that

e, For any two points p, q in M, there exists a unique geodesic γ of M joining p and q whose length is the distance from p to q.

The map exp_p will be defined in the following.

Exponential map

Throughout this thesis, we shall only consider complete Riemannian manifold M. At each point p of M, the exponential map $exp_p : T_pM \longrightarrow M$ is well defined on all of T_pM and is surjective onto M. We recall that exp_p is defined as follows: for all $v \in T_pM$,

$$exp_p(v) = \gamma_v(1),$$

where γ_v is the unique geodesic through p with tangent vector v at p. Clearly, one has

$$\gamma_v(t) = exp_p(tv)$$

for all $v \in T_pM$ and all $t \in \mathbb{R}$.

1.1.2 Curvature tensor

For each pair tangent vectors $u, v \in T_pM$, the curvature operator $R(u, v) : T_pM \longrightarrow T_pM$ is given by

$$R(u,v)w = \nabla_u \nabla_v w - \nabla_v \nabla_u w - \nabla_{[u,v]} w, \quad \forall w \in T_pM$$

where ∇ is the Riemannian connection and $[\ ,\ ]$ is the Lie bracket. By checking linearity over functions, we observe that $R(u, v)w$ is a $(1, 3)$-type tensor on M.

The following lemma is well-known.

Lemma 1.1.2 *For vectors u,v,w,z, we have*

(a) $R(u, v)w = -R(v, u)w$.

(b) $R(u, v)w + R(v, w)u + R(w, u)v = 0$.

(c) $< R(u, v)w, z >= - < R(u, v)z, w >$.

(d) $< R(u, v)z, w >=< R(z, w)u, v >$.

Condition b is called the first Bianchi identity.

The sectional curvature of the plane π spanned by a pair of tangent vectors

$u, v \in T_pM$ is defined by

$$K(\pi) = K(u \wedge v) = \frac{< R(u,v)v, u >}{\|u \wedge v\|^2}$$

where $\|u \wedge v\|^2 =< u,u >< v,v > -< u,v >^2$

It is easy to check that $K(\pi)$ is independent of the choice of basis of π

A Riemannian manifold M is called a space of constant sectional curvature, or a space form if $K(u \wedge v) = c \equiv constant$ for all linearly independent $u, v \in T_pM$ and for each $p \in M$. A space form is called spherical, flat, or hyperbolic, depending on whether $c > 0, = 0, < 0$.

For any Riemannian manifold M, the trace of $R(u,v)v$ is called the Ricci curvature of v, denoted $Ric(v,v)$.

In local coordinates, let $v = \sum_{i=1}^{n} v^i \frac{\partial}{\partial x^i} \in T_pM$, then

$$Ric(v,v) = g^{jl} < R(v, \frac{\partial}{\partial x^j})\frac{\partial}{\partial x^l}, v >$$

and the Ricci tensor is

$$R_{ik} = g^{il} R_{ijkl}$$

where g_{ij} is the Riemannian metric of M.

Furthermore, the scalar

$$R = g^{ik} R_{ik}$$

is called the scalar curvature of M with metric g_{ij}.

From the definition of $K(\pi)$, if the sectional curvature of M is nonpositive, then $R(u,v)$ is negative semidefinite for each v.

1.1.3 Holonomy

Let M be a Riemannian manifold and p be a point in M. For each finitely broken differentiable curve $\alpha : [0, l] \longrightarrow M$ with $\alpha(0) = \alpha(l) = p$, one defines a map $P_\alpha : T_pM \longrightarrow T_pM$ such that $P_\alpha(v)$ is obtained from parallel translation of $v \in T_pM$ along α back to p.

For two finitely broken differentiable curves α, β, one defines the operation $P_\beta \circ P_\alpha$ to be $P_{\beta\circ\alpha}$. (i.e. we perform P_α and then P_β.)

Now, consider the set,

$$\Phi_p = \{P_\alpha : T_pM \longrightarrow T_pM | \; \textit{all closed curves } \alpha \textit{ starting at } p\},$$

Φ_p forms a subgroup of the orthogonal group in T_pM. The set Φ_p is called the holonomy group at p.

1.2 Simply-connected Manifold of Nonpositive Sectional Curvature

We first study the geometric properties of a simply-connected manifold of nonpositive curvature. In later sections, we shall study a Riemannian manifold M of nonpositive curvature by considering the action of the fundamental group $\pi_1(M)$ on the universal covering manifold $\tilde{M}$.

1.2.1 Topological structure

The Cartan-Hadamard Theorem gives us some information about the topological structure of $\tilde{M}$. It states that:

Theorem 1.2.1 (Cartan-Hadamard) *Let $\tilde{M}$ be complete of nonpositive curvature. Then, for any $p \in \tilde{M}$, $exp_p : T_p\tilde{M} \longrightarrow \tilde{M}$ is a covering map. Hence the universal covering space of $\tilde{M}$ is diffeomorphic to $\mathbb{R}^n$.*

Suppose $\tilde{M}$ is a simply-connected, complete Riemannian manifold of nonpositive curvature. By Cartan-Hadamard Theorem, the map $exp_p : T_p\tilde{M} \longrightarrow \tilde{M}$ is a diffeomorphism for all $p \in \tilde{M}$. In particular,

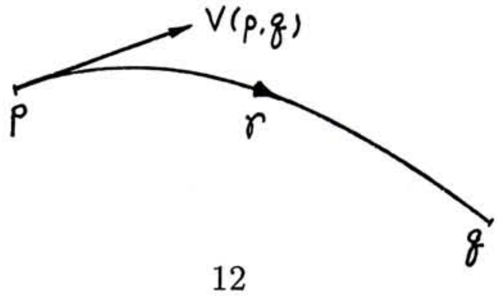
$$\tilde{M} \cong \mathbb{R}^n$$

where $n = \dim(T_p\tilde{M})$.

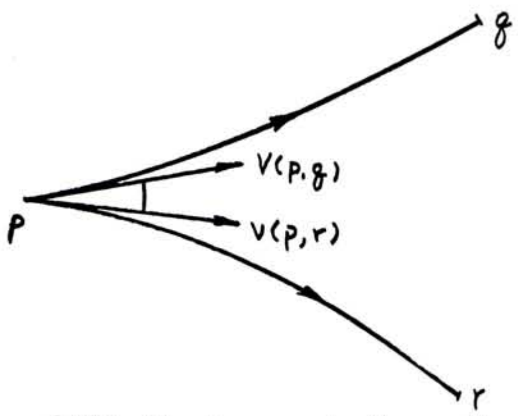
The nonpositive sectional curvature of $\tilde{M}$ implies that the map exp_p is one-one. Therefore, there exists a unique geodesic with unit speed joining any two points of $\tilde{M}$.

Notation. Let p, q, r be three distinct points in $\tilde{M}$. Let $\gamma_{pq} : [0, l] \longrightarrow \tilde{M}$ such that $\gamma_{pq}(0) = p$ and $\gamma_{pq}(l) = q$ where $l = d(p, q)$ and denote

$$V(p, q) = \gamma'_{pq}(0).$$



Let $\angle_p(q,r) = \angle(V(p,q), V(p,r))$ be the angle subtended at p by q and r.



For the remainder of this thesis, we shall use the following notations to indicate some specified terms.

$$\begin{aligned}
\tilde{M} &= \textit{a simply-connected, complete Riemannian manifold} \\
&\quad\ \textit{of nonpositive curvature.} \\
S\tilde{M} &= \textit{the unit tangent bundle of } \tilde{M}. \\
K(M) &= \textit{the sectional curvature of a Riemannian manifold } M \\
&\quad\ \textit{for all plane sections at all points of } M. \\
\gamma &= \textit{a geodesic of a Riemannian manifold } M \textit{ with unit speed.}
\end{aligned}$$

1.2.2 Basic geometric properties

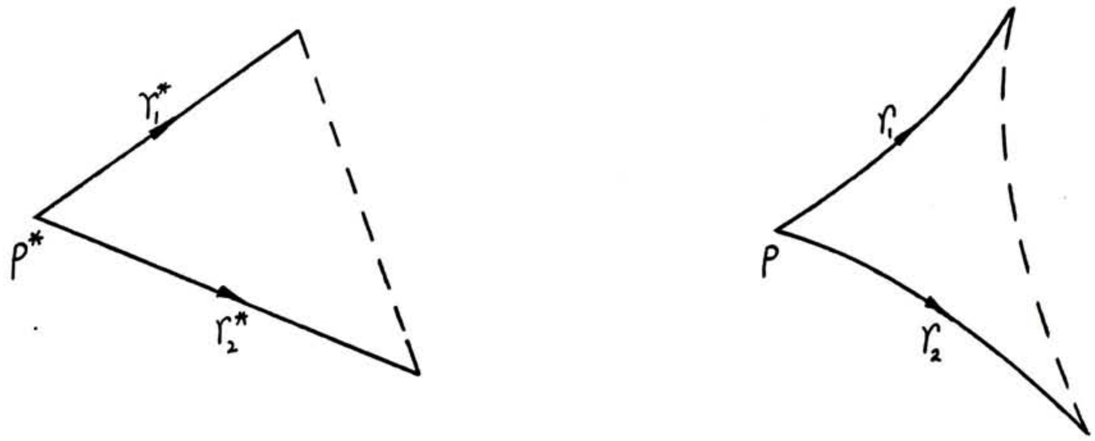
$\tilde{M}$ not only has the same topological and differentiable structure as a Euclidean space but also has some geometric properties of a Euclidean space. We shall compare the properties of $\tilde{M}$ with that of $\mathbb{R}^n$ where $n = \dim(\tilde{M})$.

Theorem 1.2.2 (Toponogov) *Let c be a nonpositive constant. Let M^* be a complete, simply-connected manifold with $K(M^*) = c$. Let $\tilde{M}$ be a complete, simply-connected manifold with $K(\tilde{M}) \leq c$. Let γ_i and γ_i^*, $i = 1, 2$, be*

geodesics starting at $p \in \tilde{M}$ *and* $p^* \in M^*$ *respectively.*
If $\angle_p(\gamma_1'(0), \gamma_2'(0)) = \angle_{p^*}(\gamma_1^{*\prime}(0), \gamma_2^{*\prime}(0))$, *then*

$$d(\gamma_1(s), \gamma_2(t)) \geq d(\gamma_1^*(s), \gamma_2^*(t)) \qquad \textit{for all } s, t \in \mathbb{R}^+.$$

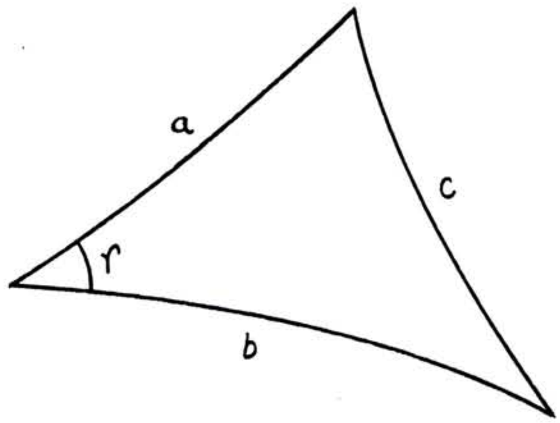
$K(M^*) = c$ $\qquad\qquad$ $K(\tilde{M}) \leq c$



The proof of this theorem can be found in [CE, chapter 2].

Intuitively, the above result says, that if $K(\tilde{M}) \leq c \leq 0$, then the geodesics starting from a point $p \in \tilde{M}$ diverge faster than in M^* with $K(M^*) = c$.
If $c = 0$, then we can obtain the law of cosine of $\tilde{M}$ by comparison with Euclidean space.

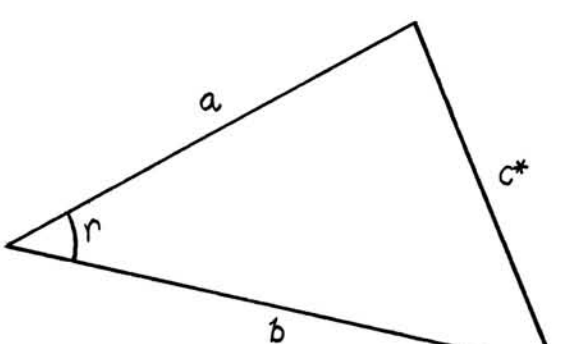
Theorem 1.2.3 (Law of Cosine) *Let $\triangle$ be a triangle in a space $\tilde{M}$ with $K(\tilde{M}) \leq 0$ as shown in the figure.*



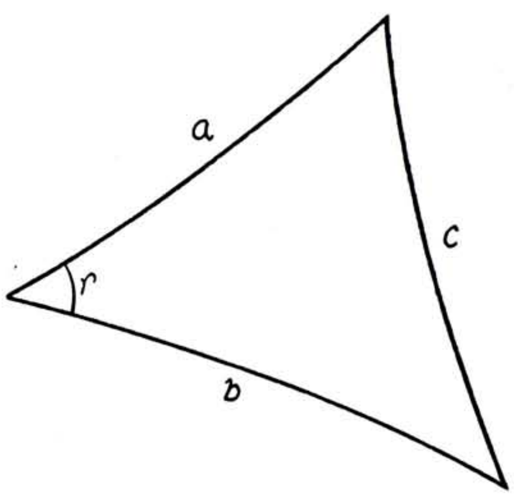
Then $$c^2 \geq a^2 + b^2 - 2abcos\gamma.$$

Proof : By theorem 1.2.2, let $K(M^*) = 0$, then we compare $\triangle$ with a triangle $\triangle_1$ in $\mathbb{R}^n$ where $n = \dim(\tilde{M})$

$\triangle_1$



$\triangle$



Therefore, we obtain $c \geq c^*$ which implies

$$c^2 \geq (c^*)^2$$

By law of cosine of $\mathbb{R}^n$,

$$(c^*)^2 = a^2 + b^2 - 2abcos\gamma.$$

We have

$$c^2 \geq a^2 + b^2 - 2abcos\gamma.$$

$\square$

Corollary 1.2.4 *If A, B, C are three noncollinear points in $\tilde{M}$, then the geodesic triangle they determine with sides of length a, b, c and opposite angles α, β, γ satisfy the following inequalities:*

(i) $c \leq bcos\alpha + acos\beta$.

(ii) $1 - cos\gamma \leq \frac{c^2}{2ab}$.

Proof : (i) By law of cosine,

$$a^2 \geq b^2 + c^2 - 2bccos\alpha$$
$$b^2 \geq a^2 + c^2 - 2accos\beta$$

Then

$$b^2 \geq b^2 + 2c^2 - 2accos\beta - 2bccos\alpha$$

Hence

$$acos\beta + bcos\alpha \geq c$$

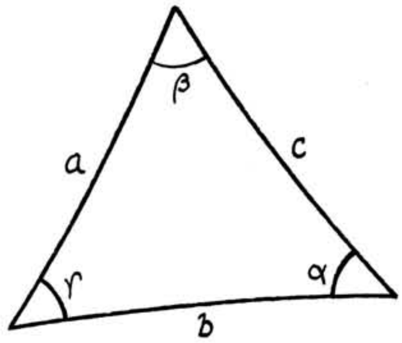
for $c \neq 0$.

(ii) By law of cosine, $c^2 \geq a^2 + b^2 - 2abcos\gamma = (a-b)^2 + 2ab(1-cos\gamma) \geq 2ab(1-cos\gamma)$ Therefore, we obtain

$$1 - cos\gamma \leq \frac{c^2}{2ab}.$$

□

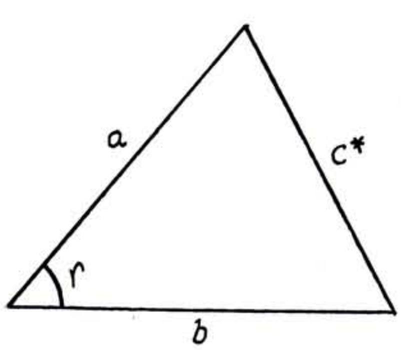
Corollary 1.2.5 (Law of angles sum) *Let $\triangle$ be a geodesic triangle in a space $\tilde{M}$ with $K(\tilde{M}) \leq 0$. Then the sum of the interior angles of $\triangle$ is at most π*

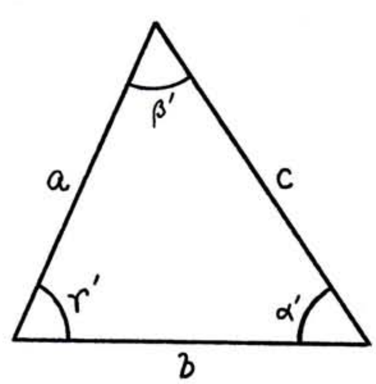


i.e. $\alpha + \beta + \gamma \leq \pi$

Proof : We draw two triangles $\triangle_1$ and $\triangle_2$ in $M^* = \mathbb{R}^n$ with the properties as shown in the figure.

$\triangle_1$ $\triangle_2$





Applying Toponogov theorem to triangles $\triangle$ and $\triangle_1$, we have,

$$c^2 \geq (c^*)^2.$$

By law of cosine in $\triangle_1$ and $\triangle_2$,

$$\begin{aligned}(c^*)^2 &= a^2 + b^2 - 2abcos\gamma \\ c^2 &= a^2 + b^2 - 2abcos\gamma'.\end{aligned}$$

Therefore,

$$a^2 + b^2 - 2abcos\gamma' \geq a^2 + b^2 - 2abcos\gamma,$$

hence,

$$\gamma' \geq \gamma$$

Similarly, $\alpha' \geq \alpha$ and $\beta' \geq \beta$.

Thus, $$\alpha + \beta + \gamma \leq \alpha' + \beta' + \gamma' \leq \pi.$$ □

Proposition 1.2.6 *Let $p \in \tilde{M}$, $\{r_n\}$ and $\{q_n\}$ be sequences in $\tilde{M}$ such that $d(r_n, p) \to \infty$ and $d(r_n, q_n) \leq k$ for all $n \in \mathbb{N}$ and some $k \in \mathbb{R}^+$.*
Then $\angle_p(q_n, r_n) \to 0$ as $n \to \infty$.

Proof : From (ii) of corollary 1.2.4,

$$\lim_{n\to\infty}(1 - cos\gamma) \leq \lim_{n\to\infty} \frac{d^2(r_n, q_n)}{2d(r_n, p)d(q_n, p)} = 0$$

Therefore, $\lim_{n\to\infty} cos\gamma = 1$ where $\gamma = \angle_p(q_n, r_n)$.

Hence, $\gamma \to 0$ as $n \to \infty$. □

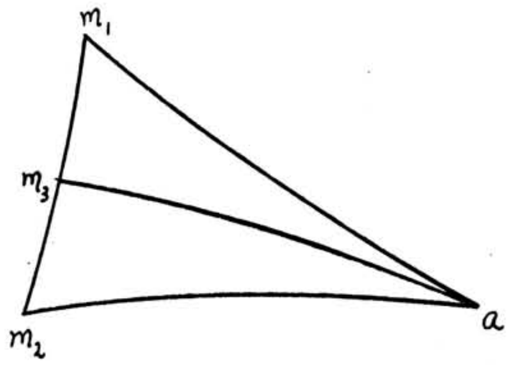
As an application, one can prove the following theorem on fixed points of isometries.

Theorem 1.2.7 (Cartan) *Let Γ be a group of isometries of a space $\tilde{M}$ such that the orbit $\Gamma(p)$ is bounded in $\tilde{M}$ for some point p of $\tilde{M}$. Then Γ has a fixed point in $\tilde{M}$.*

Proof : Let $A = \overline{\Gamma(p)}$, the closure of $\Gamma(p)$ in $\tilde{M}$. Clearly, A is Γ-invariant. By Hopf-Rinow theorem, A is compact.
Now let $r : \tilde{M} \longrightarrow \mathbb{R}$ be defined by

$$r(q) = \max\{d(q,a) : a \in A\}.$$

Since A is compact, there is a point $m \in A$ such that $r(m) \leq r(q)$ for all $q \in A$. Therefore, we shall show that r attains a minimum value r_0 at a unique point $m \in A$. Suppose that there exists m_1 and m_2 in A such that $r(m_1) = r(m_2) = r_0$.
Let m_3 be the mid-point of the geodesic segment from m_1 to m_2 as shown in the figure.



Pick $a \in A$, then we consider the two geodesic triangles $\triangle am_1m_3$ and $\triangle am_2m_3$. If we assume, $\angle_{m_3}(a, m_1) \geq \frac{\pi}{2}$, then, by the law of cosine, in $\triangle am_1m_3$,

$$d(a,m_1) \geq d(a,m_3)$$

Therefore,

$$d(a,m_3) \leq \max\{d(a,m_1), d(a,m_2)\} \leq r_0 \qquad \forall a \in A$$

Hence,

$$r(m_3) \leq r_0.$$

This contradicts the definition of r_0.

Consequently, by uniquenss of m, Γ fixes a point $m \in A$. $\square$

1.2.3 Examples of nonpositively curved manifold

Example 1. Euclidean space $\mathbb{R}^n$ with its canonical inner product. Here $K \equiv 0$. The geodesic are the straight lines and the isometries are all products $R \circ T$, where $R \in O(n)$, the orthogonal group, and T is a translation of $\mathbb{R}^n$.

Example 2. Hyperbolic spaces H^n with $K(H^n) \equiv -1$. We describe two models:

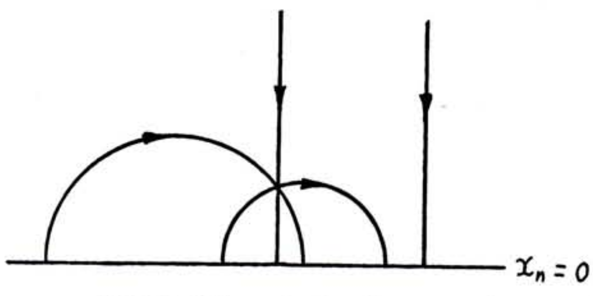
(a) Upper half-space model.

Let $H^n = \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n : x_n > 0\}$.

Let H^n be given the Riemannian metric

$$ds^2 = \frac{{dx_1}^2 + \dots + {dx_n}^2}{x_n^2}$$

which is conformally equivalent to the usual Euclidean metric on H^n. The geodesics of H^n are either the "vertical" lines or Euclidean circular arcs in $\mathbb{R}^n$ that are orthogonal to the hyperplane $x_n = 0$ and have constant speed in the hyperbolic metric.



If $n = 2$, then the group $SL(2, \mathbb{R})$ acts by isometries on H^2 by

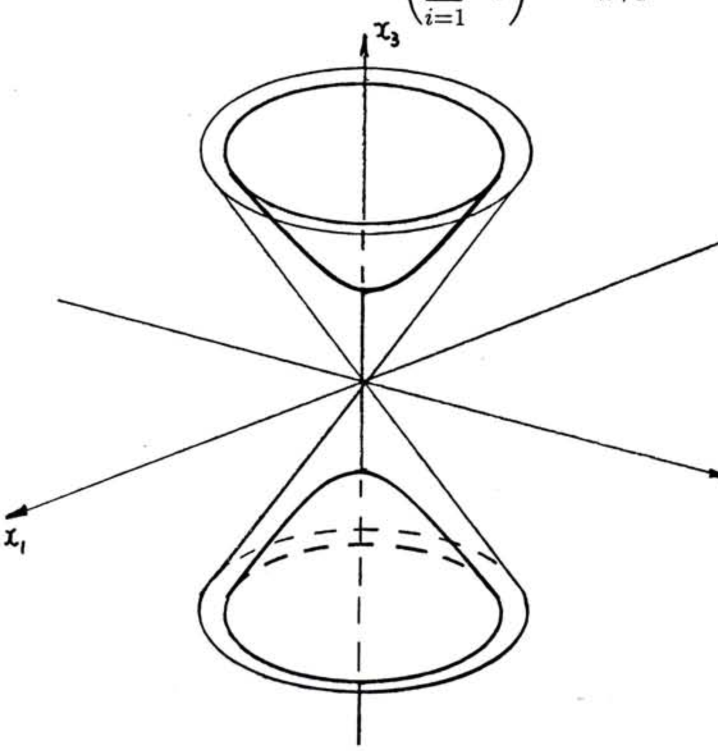
$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}(z) = \frac{az+b}{cz+d}$$

where a point $(x, y) \in H^2$ is identified with the complex number $z = x + yi$. Clearly, the matrices A and $-A$ in $SL(n, \mathbb{R})$ induce the same action on H^2. In fact, $I_0(H^2) = PSL(2, \mathbb{R}) = SL(2, \mathbb{R})/\{\pm I\}$ where $I_0(H^2)$ is the connected component of the isometry group of H^2 that contains the identity.

(b) Hyperboloid model.

Let

$$H^n = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R} : \left(\sum_{i=1}^{n} x_i^2\right) - x_{n+1}^2 = -1, x_{n+1} > 0\}$$



Let $SO(n, 1)$ denote the subgroup of $SL(n+1, \mathbb{R})$ which leaves invariant the bilinear form ϕ:

$$(x, y) \longrightarrow \left(\sum_{i=1}^{n} x_i y_i\right) - x_{n+1} y_{n+1}$$

where $x = (x_1, \dots, x_{n+1})$ and $y = (y_1, \dots, y_{n+1})$ are points in $\mathbb{R}^{n+1}$.

Let $v = (v_1, \dots, v_{n+1}) \in T_p H^n$ for some $p = (x_1, \dots, x_{n+1}) \in H^n$, then $v_1, \dots, v_{n+1}$ must satisfy

$$\sum_{i=1}^{n} x_i v_i - x_{n+1} v_{n+1} = 0.$$

Now, we consider the restriction of ϕ on $T_p H^n$, we obtain,

$$\begin{aligned}
& \sum_{i=1}^{n} {v_i}^2 - v_{n+1}^2 \\
= \;& \sum_{i=1}^{n} {v_i}^2 - \left(\frac{\sum_{i=1}^{n} x_i v_i}{x_{n+1}}\right)^2 \\
= \;& \frac{1}{x_{n+1}^2}\left(x_{n+1}^2 \sum_{i=1}^{n} {v_i}^2 - \left(\sum_{i=1}^{n} x_i v_i\right)^2\right) \\
= \;& \frac{1}{x_{n+1}^2}\left(\left(\sum_{i=1}^{n} {x_i}^2 + 1\right)\left(\sum_{i=1}^{n} {v_i}^2\right) - \left(\sum_{i=1}^{n} x_i v_i\right)^2\right) \\
= \;& \frac{1}{x_{n+1}^2}\left(\left(\sum_{i=1}^{n} {x_i}^2\right)\left(\sum_{i=1}^{n} {v_i}^2\right) - \left(\sum_{i=1}^{n} x_i v_i\right)^2 + \sum_{i=1}^{n} {v_i}^2\right) \\
\geq \;& 0.
\end{aligned}$$

Therefore, ϕ is positive definite on $T_p H^n$, for all $p \in H^n$. We thus obtain a Riemannian inner product ϕ on H^n. Since, $SO(n, 1)$ leaves ϕ invariant, then it forms an isometry subgroup of H^n. Moreover, the identity $I \in SL(n+1, \mathbb{R})$

is contained in $SO(n,1)$. Therefore, $SO(n,1) = I_0(H^n)$.

The geodesics of H^n that start at $e_{n+1} = (0, \ldots, 0, 1)$ are the intersections of H^2 with 2-dimensional linear subspaces in $\mathbb{R}^{n+1}$ that contain the vector e_{n+1}. Since $SO(n,1)$ acts transitively on H^n, then all other geodesics of H^n are images of the geodesics just described under $SO(n,1)$.

Example 3. The space $\tilde{M}_n$ consisting of all positive definite symmetric $n \times n$ matrices of determinant 1. The space $\tilde{M}_n$ will be discussed below in chapter 2.

1.2.4 Convexity properties

As in Euclidean space, a subset W in $\tilde{M}$ is called convex, if for $p, q \in W$ there is a unique geodesic from p to q in $\tilde{M}$ and this geodesic is contained in W. The definition makes sense since any two points in $\tilde{M}$ can be joined by a unique geodesic in $\tilde{M}$.

A continuous function $h : \mathbb{R} \to \mathbb{R}$ is called convex, if for $a < b$ and $\lambda \in [0,1]$, we have,

$$h(\lambda a + (1-\lambda)b) \leq \lambda h(a) + (1-\lambda)h(b)$$

One defines a continuous function $f : \tilde{M} \longrightarrow \mathbb{R}$ to be convex if $f \circ \gamma : \mathbb{R} \longrightarrow \mathbb{R}$ is convex for any geodesic γ in $\tilde{M}$.

In case $h : \mathbb{R} \longrightarrow \mathbb{R}$ is C^2, h is convex if and only if $h'' \geq 0$ everywhere. Hence, a C^2 function $f : \tilde{M} \longrightarrow \mathbb{R}$ is convex if and only if $(f \circ \gamma)'' \geq 0$ for all geodesic γ in $\tilde{M}$.

If strict inequality hold in the above inequalities, one calls the functions strictly convex.

We first observe a simple relation between convex functions and convex sets in $\tilde{M}$.
Namely, if $f : \tilde{M} \longrightarrow \mathbb{R}$ is a convex function, then for any $a \in \mathbb{R}$, the level subset

$$\tilde{M}^a = \{p \in \tilde{M} : f(p) \leq a\}$$

is a closed convex subset of $\tilde{M}$.
Indeed, let $\gamma : [0,1] \longrightarrow \tilde{M}$ be a geodesic segment with $(f \circ \gamma)(0) \leq a$ and $(f \circ \gamma)(1) \leq a$. The convexity of $f \circ \gamma$ implies that $(f \circ \gamma)$ attains maximum value at the boundary of $[0,1]$. Hence, γ is contained in $\tilde{M}^a$.

It is interesting to compare the convexity properties of Euclidean space and $\tilde{M}$. As we shall also see, the following properties are useful.

Let A be a closed convex subset of a space $\tilde{M}$. If $p \in A$, then $x \in T_p\tilde{M}$ is said to be tangent to A provided x is the initial velocity of some curve α that is initially in A (i.e. $x = \alpha'(0)$ where $\alpha([0,\epsilon]) \subset A$). If $m \in \tilde{M}$, a perpendicular from m to A is a geodesic segment $\gamma : [0,1] \longrightarrow \tilde{M}$ such that

(1) $\gamma(0) = m$

(2) $\gamma(1) \in A$

(3) $< \gamma'(1), x > \geq 0$ for every tangent x to A at $\gamma(1)$.

Note that if $m \in A$, then the constant geodesic segment at m is the unique perpendicular from m to A. In fact, if γ is a geodesic loop at $m \in A$, then γ lies in A, so $\gamma(1-t)$ is initially in A. Hence, $< \gamma'(1), -\gamma'(1) > \geq 0$ and γ is constant.

Proposition 1.2.8 *Let A be a closed convex subset of a space $\tilde{M}$. Then, for each $p \in \tilde{M}$, there exists a unique point $m \in A$ such that*

$$d(p,m) \leq d(p,q) \qquad\qquad \forall q \in A$$

Proof : Since A is closed, if $p \in \tilde{M}$, then there is a point $m \in A$ nearest to p. Let $\gamma_p : [0,1] \longrightarrow \tilde{M}$ the geodesic segment from p to m. Since γ_p is a shortest geodesic from p to A, we only need to show that γ_p is the unique shortest geodesic segment from p to A.

Let $\alpha_p : [0,1] \longrightarrow \tilde{M}$ be another shortest geodesic segment from p to $m' \in A$ and α_p is perpendicular to A. The property, $K(\tilde{M}) \leq 0$, implies $\gamma_p(1) \neq \alpha_p(1)$.

Let $\beta : [0,1] \longrightarrow \tilde{M}$ be the geodesic segment such that

$$\beta(0) = \gamma_p(1), \beta(1) = \alpha_p(1)$$

By convexity of $A, \beta \in A$.

It follows that

$$\angle_{\beta(0)}({\gamma_p}'(1), \beta'(0)) \geq \tfrac{\pi}{2} \text{ and } \angle_{\beta(1)}({\alpha_p}'(1), \beta'(1)) \geq \tfrac{\pi}{2}$$

Therefore, the angles sum of $\triangle p\beta(0)\beta(1) > \pi$. This contradiction shows that $\gamma_p = \alpha_p$.

Hence, $m = \gamma_p(1) = \alpha_p(1) = m'$. $\square$

The point m is called the foot point of p on A.

Proposition 1.2.9 *Let A be a closed convex subset of a space $\tilde{M}$. Then the function $f_A : \tilde{M} \longrightarrow \mathbb{R}$ given by*

$$f_A(p) = d^2(p, m) = d^2(p, A)$$

is a continuous convex function where m is the foot point of p. Moreover, f_A is strictly convex on $\tilde{M} - A$ if $K(\tilde{M}) < 0$.

Proof : For each $p \in \tilde{M}$, there is a unique perpendicular γ_p from p to A. By definition of exponential map and Cartan-Hadamard Theorem,

$$f_A(p) = d^2(p, m) = |exp_p{}^{-1}(m)|^2.$$

Therefore, f_A is C^∞ and $f_A(p) = |\gamma_p|^2$.
Let $0 \neq y \in T_p\tilde{M}$ and let β be the geodesic with initial velocity y. Let r be the rectangle given by $r(t, v) = \gamma_{\beta(v)}(t)$. Let Y be the transverse vector field of r (so $Y = \frac{\partial r}{\partial v}$ and $Y(0) = y$).
Then $(\nabla^2 f_A)(y, y)$ is the second variation of the squared arc length L^2 for the rectangle r.
Therefore, we have,

$$\begin{aligned} (\nabla^2 f_A)(y, y) &= L''(0) \\ &= < \nabla_Y Y, \gamma_p{}' > |_0^1 + \int_0^1 \{ \|Y^{\perp\prime}\|^2 - < R_{\gamma_p' Y^\perp}\gamma_p{}', Y^\perp > \} dt \end{aligned}$$

where $Y^\perp$ is the orthogonal complement of Y.
Since the first transverse curve β is a geodesic, it follows that

$$< \nabla_{Y(0)} Y(0), \gamma_p{}'(0) >= 0.$$

Also, the last transverse curve lies in A which is normal to $\gamma_p{}'(1)$, therefore,

$$< \nabla_{Y(1)} Y(1), \gamma_p{}'(1) >= 0.$$

Hence,

$$\begin{aligned}(\nabla^2 f_A)(y,y) &= \int_0^1 \{\|Y^{\perp\prime}\|^2 - < R_{\gamma_p' Y^\perp}\gamma_p', Y^\perp >\}dt \\ &= \int_0^1 \{\|Y^{\perp\prime}\|^2 - K(\gamma_p' \wedge Y^\perp)\|\gamma_p' \wedge Y^\perp\|^2\}dt \\ &\geq 0.\end{aligned}$$

Since $K(\tilde{M}) \leq 0$, then f_A is convex.

To show f_A is strictly convex on $\tilde{M} - A$ for $K(\tilde{M}) < 0$, in fact, we only to show that $(\nabla^2 f_A)(y,y) = 0$ for $y \neq 0$ if and only if y is tangent to A.

If $K(\tilde{M}) < 0$ and $(\nabla^2 f_A)(y,y) = 0$, we must have $\|Y^{\perp\prime}\| = 0$, so Y is parallel along γ_p, and in particular is never zero.

Also we have $Y^\perp \wedge \gamma_p' = 0$; hence $\gamma_p'(1) = \lambda Y^\perp(1) \in T_{\gamma_p(1)}A$. But $\gamma_p'(1)$ is normal to A, hence $\gamma_p'(1) = 0$. It follows that $y \in T_pA$ since γ_p is the constant curve $\gamma_p = p \in A$. □

Proposition 1.2.10 *If ϕ is any isometry of a space $\tilde{M}$, then the function $d_\phi^2 : \tilde{M} \longrightarrow \mathbb{R}$ given by $d_\phi^2(p) = d^2(p, \phi p)$ is a C^∞ convex function on $\tilde{M}$. The function d_ϕ^2 has a positive minimum value at a point p of $\tilde{M}$ if and only if $(\phi \circ \gamma)(t) = \gamma(t+w)$ for all $t \in \mathbb{R}$, where $w = d(p, \phi p)$ and $\gamma(t)$ is the unique geodesic with $\gamma(0) = p$ and $\gamma(w) = \phi(p)$. (i.e. ϕ translates γ by an amount w.)*

A detailed proof which we omit here, can be found in [BO, proposition 4.2].

1.2.5 Points at infinity for $\tilde{M}$

Definition 1.2.11 *Two unit speed geodesics γ and σ of $\tilde{M}$ (i.e. γ, σ :*

$[0, \infty] \longrightarrow \tilde{M}$) *are called asymptotic if there exists* $c \in \mathbb{R}$ *with*

$$d(\gamma(t), \sigma(t)) \leq c$$

for all $t \geq 0$.

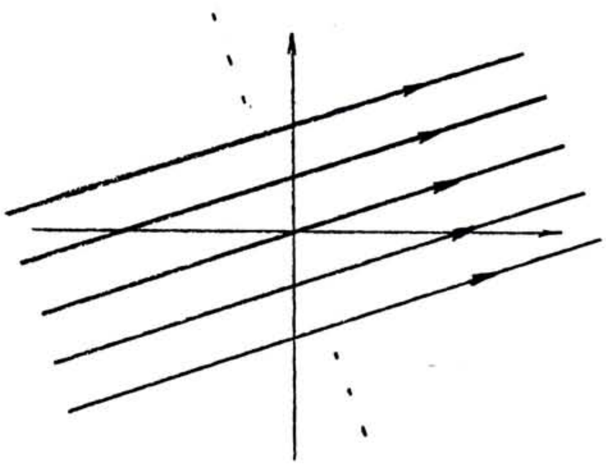
The asymptotic relation is clearly an equivalence relation on the space of unit speed geodesics in $\tilde{M}$.

Definition 1.2.12 *A point at infinity of* $\tilde{M}$ *is an equivalence class of asymptotic geodesics of* $\tilde{M}$.

The set of all points at infinity for $\tilde{M}$ is denoted by $\tilde{M}(\infty)$. $\gamma(\infty)$ denote the equivalence class represented by a geodesic γ of $\tilde{M}$ and $\gamma(-\infty)$ denote the equivalence class represented by the geodesic $\gamma^{-1} : t \longrightarrow \gamma(-t)$.

Examples

(1) If $\tilde{M} = \mathbb{R}^n$, the Euclidean space with its usual inner product, then asymptotic geodesics are parallel lines and a point at infinity is a family of parallel lines.



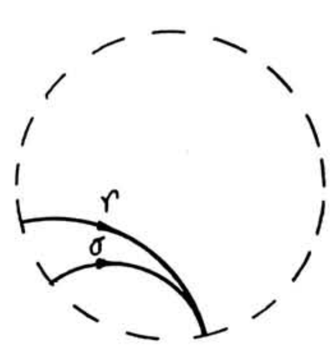
(2) If $\tilde{M} = H^2$ is the hyperbolic plane, represented as the open unit disk

$x^2 + y^2 < 1$ with the metric

$$ds^2 = \frac{2(dx^2 + dy^2)}{1 - x^2 - y^2},$$

then the unparameterized geodesics of H^2 are either the Euclidean circles that are orthogonal to the unit circle $S^1 = \{(x, y) \in \mathbb{R} : x^2 + y^2 = 1\}$ or the straight lines through the origin in $\mathbb{R}^2$.

Then $\gamma(t), \sigma(t)$ are asymptotic if and only if they converge to the same point of the boundary of S^1 in the forward direction.



Hence $H^2(\infty)$ may be identified with S^1 topologically.

Proposition 1.2.13 *Let γ be any geodesic of $\tilde{M}$. Then for each point $p \in \tilde{M}$, there exists a unique geodesic σ of $\tilde{M}$ such that $\sigma(0) = p$ and σ is asymptotic to γ.*

Proof : Since $\tilde{M}$ is complete, then $\gamma(t)$ is defined for all $t \in \mathbb{R}$. Let $\{\sigma_n\}$ be a sequence of geodesics such that they join the point p to $\gamma(n)$ for all $n \in \mathbb{Z}^+$ (i.e. $\sigma_n(0) = p$ and $\sigma_n(t_n) = \gamma(n)$ where $t_n = d(p, \gamma(n))$). Let $A = \gamma(\mathbb{R})$, then by convexity of the map $p \longrightarrow d^2(p, A)$ and $\sigma_n(t_n) \in A$, we have

$$d(\sigma_n(t), A) \leq d(p, A) \qquad \text{for } 0 \leq t \leq t_n.$$

If σ is a unit speed geodesic such that $\sigma(0) = p$ and $\sigma'(0)$ is a cluster point of the sequence of unit vectors $\{\sigma_n'(0)\}$, then by continuity,

$$d(\sigma(t), A) \leq d(p, A) \qquad \text{for all } t \geq 0.$$

Let $c = d(p, A)$, then

$$d(\sigma(t), \gamma(t)) \leq c \qquad \text{for all } t \geq 0.$$

Therefore, σ is asymptotic to γ.

Now, suppose there are two geodesics $\sigma_1(t)$ and $\sigma_2(t)$ such that $\sigma_1(0) = \sigma_2(0) = p$ and they are asymptotic to γ.
Therefore, $\sigma_1(t)$ is asymptotic to $\sigma_2(t)$. Hence,

$$d(\sigma_1(t), \sigma_2(t)) \leq c \qquad \text{for some } c > 0 \text{ and for all } t \geq 0$$

Then by law of cosine,

$$d^2(\sigma_1(t), \sigma_2(t)) \geq |\sigma_1(t)|^2 + |\sigma_2(t)|^2 - 2|\sigma_1(t)||\sigma_2(t)|cos\theta.$$

With loss of generality, suppose $|\sigma_1(t)|^2 = |\sigma_2(t)|^2$, then

$$d^2(\sigma_1(t), \sigma_2(t)) \geq 2|\sigma_1(t)|^2(1 - cos\theta) \geq 2|\sigma_1(t)|^2.$$

Hence,

$$d^2(\sigma_1(t), \sigma_2(t)) \to \infty \text{ as } n \to \infty$$

This contradicts the asymptotic relation of $\sigma_1(t)$ and $\sigma_2(t)$.

Therefore, $\sigma_1 = \sigma_2$. □

Corollary 1.2.14 *For each pair $p \in \tilde{M}, x \in \tilde{M}(\infty)$, there exists a unique geodesic $\gamma = \gamma_{px}$ such that $\gamma_{px}(0) = p$ and $\gamma_{px}(\infty) = x$ (i.e.γ_{px} starts at p and belongs to the asymptotic class x.).*

Remark . From the above result, $\tilde{M}(\infty)$ may be identified with $(n-1)$-sphere S^{n-1}. More precisely, $\tilde{M}(\infty)$ can be identified with the $(n-1)$-sphere of unit vectors at any point $p \in \tilde{M}$.

The cone topology

Let $\bar{M} = \tilde{M} \cup \tilde{M}(\infty)$. One defines a natural topology that makes $\bar{M}$ homeomorphic to the closed unit ball in $\mathbb{R}^n$ and $\tilde{M}(\infty)$ homeomorphic to the S^{n-1}. One uses the notion of angle to measure the proximity of two points at infinity.

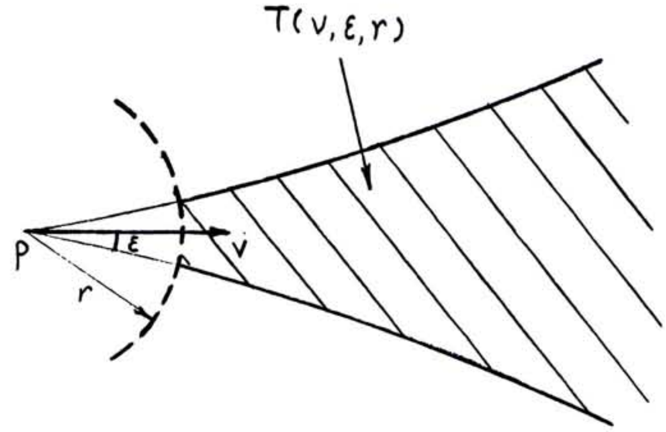
Let p be a point of $\tilde{M}$. If x and y are points of $\bar{M} = \tilde{M} \cup \tilde{M}(\infty)$ that are distinct from p, then the angle subtended by x, y at p is

$$\angle_p(x, y) = \angle(\gamma'_{px}(0), \gamma'_{py}(0)).$$

Definition 1.2.15 *Let $v \in T_p\tilde{M}$ and let ϵ be a number,$0 \leq \epsilon \leq \pi$. Then for any number $r \geq 0$, one calls*

$$T(v,\epsilon,r) = \{q \in \bar{M} : \angle_p(\gamma_v(\infty), q) \leq \epsilon\} - \{q \in \tilde{M} : d(p,q) \leq r\}$$

the truncated cone of vertex p, axis v, angle ϵ and radius r.



For fixed point $p \in \tilde{M}$ and fixed unit vector $v \in T_p\tilde{M}$, the truncated cones $T(v,\epsilon,r)$ as ϵ and r range over all positive numbers form a neighborhood basis at x in $\bar{M} = \tilde{M} \cup \tilde{M}(\infty)$. This topology is called the cone topology of $\bar{M}$.

Proposition 1.2.16 *The cone topology τ for $\bar{M}$ satisfies the following conditions:*

1. *Closure property: the topology on $\tilde{M}$ induced by τ is the original topology of $\tilde{M}$ and $\tilde{M}$ is a dense open set of $\bar{M}$.*

2. *Geodesic extension property: if α is any geodesic of M, then its asymptotic extension is continuous.*

3. *Isometric extension property: if φ is any isometry of $\tilde{M}$, then its asymptotic extension is continuous.*

4. *Intensive property: if $x \in \tilde{M}(\infty)$, V is a neighborhood of x, and $r \geq 0$ is any positive number, then there exists a neighborhood U of x such*

that $N_r(U) = \{q \in \bar{M} : d(q,U) \leq r\} \subseteq V$. Here we have extended the metric trivially so that $d(a,b) = \infty$ if $a \neq b$ and either point lies in $\tilde{M}(\infty)$

For a discussion with references see [EO, proposition 2.9]

Proposition 1.2.17 *Let $D = \{(p,x) \in \tilde{M} \times \tilde{M}(\infty) : p \neq p\}$. Then the map $V : D \longrightarrow S\tilde{M}$ given by $(p,x) \longrightarrow V(p,x) = \gamma'_{px}(0)$ is a homeomorphism with respect to the product topology in $\tilde{M} \times \tilde{M}(\infty)$*

It is also easy to show that the map $\angle_p(q,r) = \angle(\gamma'_{pq}(0), \gamma'_{pr}(0))$ is continuous on $\{(p,q,r) \in \tilde{M} \times \bar{M} \times \bar{M} : p \neq q, p \neq r\}$.
Note that $cos(\angle_p(x,y)) =< \gamma'_{px}(0), \gamma'_{py}(0) >$.

Joining points at infinity

Definition 1.2.18 *Let p, q be two distinct points in $\tilde{M}(\infty)$. Then p, q can be joined by a geodesic of $\tilde{M}$ if there exists a geodesic γ of $\tilde{M}$ such that $\gamma(\infty) = p$ and $\gamma(-\infty) = q$.*

Examples

1. If $\tilde{M} = \mathbb{R}^n$, then each point $x \in \tilde{M}(\infty)$ can be joined to a unique point $y \in \tilde{M}(\infty)$ but the joining geodesic is not unique.
2. If $\tilde{M} = H^n$, hyperbolic space, then any two distinct points $x, y \in \tilde{M}(\infty)$ can be joined by a unique geodesic of $\tilde{M}$.

For a geodesic $\gamma : \mathbb{R} \longrightarrow \tilde{M}$, one calls $\gamma^+ = \gamma|_{[0,\infty)}$ the positive ray of γ and $\gamma^- = \gamma|_{(0,-\infty)}$ the negative ray.

Lemma 1.2.19 *Let $\tilde{M}$ be simply-connected, complete Riemannian manifold of sectional curvature $K \leq c < 0$ for some negative constant c. Let σ_1 and σ_2 be distinct geodesic rays strating at $p \in \tilde{M}$. Then there is a geodesic γ such that γ^+ is asymptotic to σ_1 and γ^- is asymptotic to σ_2.*

Proof : Suppose that σ_1 and σ_2 have unit speed. For $i \geq 1$, let γ_i be the unit speed geodesic through $\sigma_1(i)$ and $\sigma_2(i)$. Since the function $d^2(p, \cdot)$ is strictly convex, the foot $\gamma_i(t)$ of p on γ_i lies between $\sigma_1(i)$ and $\sigma_2(i)$. Let $d_i = d(p, \gamma_i(t))$ Let A_i be the area of the triangle composed of all geodesic segments from p to points γ_i between $\sigma_1(i)$ and $\sigma_2(i)$. On each such radial segment the points with distance at most d_i from p form a patch of surface. It can be shown that the area of such surface is greater than that of a Euclidean sector of angle $\theta = \angle < \sigma_1'(0), \sigma_2'(0) >$ and radius d_i. Thus $A_i > \frac{1}{2}\theta d_i^2$. On the other hand, the Gauss-Bonnet theorem implies that $A_i < \pi/|c|$, where $K \leq c < 0$. Hence d_i is bounded above as i goes to infinity. If γ_i is parameterized such that $d_i(p, \gamma_i(0))$ is bounded above, then $\{\gamma_i'(0)\}$ converges to a vector v as $n \to \infty$ and the geodesic γ with initial velocity v has the required properties. □

From this result, one concludes that if $\tilde{M}$ has sectional curvature $K \leq c < 0$ for some negative constant c, then any two distinct points x, y of $\tilde{M}(\infty)$ can be joined by a unique geodesic γ of $\tilde{M}$. The uniqueness of geodesic γ follows from the strict convexity of $p \to d^2(p, \gamma)$ on $\tilde{M} - \gamma$ given by proposition 1.2.9.

Action of isometries on $\tilde{M}(\infty)$

If ϕ is any isometry of $\tilde{M}$ and if x is a point in $\tilde{M}(\infty)$, then ϕ extends to homeomorphism of $\tilde{M}(\infty)$ with respect to the cone topology by defining

$\phi(x) = (\phi \circ \gamma)(\infty)$ where γ represents x. Also, the map $(\phi, x) \longrightarrow \phi(x)$ of $I(\tilde{M}) \times \tilde{M}(\infty) \longrightarrow \tilde{M}(\infty)$ is continuous with respect to the product topologies where $I(\tilde{M})$ is the isometry group of $\tilde{M}$.

Chapter 2

Symmetric Spaces

2.1 Symmetric Spaces of Noncompact Type

2.1.1 Symmetric diffeomorphisms

Let $\tilde{M}$ be simply-connected, complete Riemannian manifold of nonpositive curvature. Given a point p in $\tilde{M}$, it is well known that the exponential map $exp_p : T_p\tilde{M} \longrightarrow \tilde{M}$ is a diffeomorphism. One defines the geodesic symmetry $s_p : \tilde{M} \longrightarrow \tilde{M}$ by

$$s_p = exp_p \circ s \circ {exp_p}^{-1}$$

where $s(v) = -v, \forall v \in T_p\tilde{M}$.

Equivalently,

$$s_p(\gamma(t)) = \gamma(-t)$$

for all geodesics γ of $\tilde{M}$ with $\gamma(0) = p$ and for all $t \in \mathbb{R}$.

The map s_p fixes p and is a diffeomorphism of $\tilde{M}$. Clearly, $s_p \circ s_p$ =identity, for all $p \in \tilde{M}$.

One defines

$$\begin{aligned} G^* &= \textit{the group of diffeomorphisms of } \tilde{M} \textit{ generated by} \\ & \quad \textit{the geodesic symmetries } \{s_p : p \in \tilde{M}\}. \\ {G_e}^* &= \textit{the subgroup of } G^* \textit{ consisting of elements that are} \\ & \quad \textit{the product of an even number of geodesic symmetries.} \end{aligned}$$

The group G^* is called the symmetry diffeomorphism group of $\tilde{M}$.
The subgroup ${G_e}^*$, which has index 2 and hence is normal in G^*, is called the group of even symmetry diffeomorphisms.

Definition 2.1.1 *A simply-connected space $\tilde{M}$ is said to be a symmetric space if $G^* \subseteq I(\tilde{M})$ or equivalently, if for each $p \in \tilde{M}$, s_p is an isometry of $\tilde{M}$.*

Examples

1. $\mathbb{R}^n$ is a symmetric space with $K(\mathbb{R}^n) \equiv 0$.
2. H^n is a symmetric space of noncompact type(see definition 2.1.14) with $K(H^n) \equiv -1$.
3. $S^{n-1} \subseteq \mathbb{R}^n$ of radius 1 is a symmetric space of compact type(see definition 2.1.14) with $K(S^{n-1}) \equiv 1$.
4. If G is any compact Lie group and $<, >$ is any inner product on G that is invariant under left and right translations by elements of G, then G is a symmetric space with $K(G) \geq 0$. See [Mi, p.109-115] for details.
5. The space $\tilde{M}_n$ of noncompact type consisting of all $n \times n$ symmetric, positive definite matrices with determinant 1. See section 2.2.

2.1.2 Transvections in $I(\tilde{M})$

Definition 2.1.2 *Fix a point $p \in \tilde{M}$. An isometry ϕ of $\tilde{M}$ is a transvection at p if there exists a unit speed geodesic γ of $\tilde{M}$ such that*

1. $\gamma(0) = p$.

2. $(\phi \circ \gamma)(t) = \gamma(t + w)$ *for all $t \in \mathbb{R}$ and for some $w \in \mathbb{R}$.*

3. $d\phi : T_{\gamma(s)}\tilde{M} \longrightarrow T_{\gamma(s+w)}\tilde{M}$ *is parallel translation along γ from $\gamma(s)$ to $\gamma(s + w)$.*

Lemma 2.1.3 *Let γ be geodesic of $\tilde{M}$ such that $\gamma(0) = p$ and $\gamma(\tau) = q$. Then*

$$s_q s_p(\gamma(t)) = \gamma(t + 2\tau) \qquad \forall t \in \mathbb{R}.$$

For each $v \in T_{\gamma(t)}\tilde{M}, ds_q ds_p(v) \in T_{\gamma(t+2\tau)}\tilde{M}$ is the vector at $\gamma(t+2\tau)$ obtained by parallel transport of v along γ.

Proof : Let $\tilde{\gamma}(t) = \gamma(t + \tau)$, then $\tilde{\gamma}$ is a geodesic with $\tilde{\gamma}(0) = q$. It follows that

$$s_q s_p(\gamma(t)) = s_q(\gamma(-t)) = s_q(\tilde{\gamma}(-t - \tau)) = \gamma(t + 2\tau)$$

Let $v \in T_p\tilde{M}$ and let X be the parallel vector field along γ with $X(p) = v$. Since s_p is an isometry, ds_pX is also parallel. Moreover,

$$ds_pX(p) = -X(p).$$

Hence,

$$ds_pX(\gamma(t)) = -X(\gamma(-t))$$

and

$$ds_q \circ ds_p X(\gamma(t)) = X(\gamma(t + 2\tau)).$$

□

2.1.3 Symmetric spaces as coset manifolds G/K

If $G = I_0(M)$, the connceted component of the isometry group $I(M)$ that contains the identity, then G is a Lie group in the compact-open topology.

Lemma 2.1.4 *G acts transitively on $\tilde{M}$.*

Proof : Given distinct points p and q in $\tilde{M}$, let $\gamma : [0,1] \longrightarrow \tilde{M}$ be a geodesic with $\gamma(0) = p$ and $\gamma(1) = q$.
For each $t \in \mathbb{R}$, let s_t denote the geodesic symmetry $s_{\gamma(t)}$ and let $p_t = s_{t/2} \circ s_0$. Clearly, p_t is an isometry for all $t \in \mathbb{R}$. By lemma 2.1.3, p_t is a transvection at p such that $p_0 = I$ and $p_1(p) = q$. This shows that $G(p) = \tilde{M}$. □

If $\tilde{M}$ is a symmetric space, for each $p \in \tilde{M}$, let $G_p = \{g \in G : g(p) = p\}$, then G_p is a compact subgroup of G in the induced topology (Theorem 2.2 of [H,p.167])

Theorem 2.1.5 *Let $\tilde{M}$ be a symmetric space, and let $G = I_0(\tilde{M})$. Then every maximal compact subgroup K of G equals G_p for some point p in $\tilde{M}$ and all maximal compact subgroups of G are conjugate in G.*

Proof : One can show that G_p is a compact subgroup of G. Suppose that $G_p \subseteq K$, where K is a compact subgroup of G. By the Cartan fixed point theorem, there exists a point $q \in \tilde{M}$ that is fixed by K. Since G acts transitively on $\tilde{M}$, there is an element $g \in G$ such that $g(q) = p$. It follows that

$$gG_pg^{-1} \subseteq gKg^{-1} \subseteq g(G_q)g^{-1} = G_{g(q)} = G_p.$$

Hence g normalizes G_p and all inclusions above are equalities.

Therefore, $K = G_p$. $\square$

Corollary 2.1.6 *Let $\tilde{M}$ be a symmetric space, and let $G = I_0(\tilde{M})$. Let K be a maximal compact subgroup of G, and let $p \in \tilde{M}$ be a point fixed by K. Then the map*

$$\begin{aligned} \alpha \ : \ G/K &\longrightarrow \tilde{M} \\ gK &\longmapsto g(p) \end{aligned}$$

is a bijection.

Proof : It can be shown that α is one-one. Since G acts transitively on $\tilde{M}$. For all $q \in \tilde{M}$, there exists $g_1 \in G$ but $g_1 \notin K$ such that $g_1(p) = q$

Therefore, $\alpha(g_1K) = g_1(p) = q$ and hence α is onto. $\square$

From the above discussion, $\tilde{M}$ is diffeomorphic to the coset manifold G/K under the map $\phi : G/K \longrightarrow \tilde{M}$ given by

$$\phi(gK) = g(p).$$

If $<,>$ is the metric on $\tilde{M}$ and if $\phi^* <,>$ denotes the pullback metric on G/K that makes ϕ an isometry, then $\phi^* <,>$ is left invariant; that is, the transformation $\zeta(g) : g'H \longrightarrow gg'H$ is an isometry of $(G/K, \phi^* <,>)$.

2.1.4 Metric on $T_p\tilde{M}$ and the adjoint representation of Lie group

For each $p \in \tilde{M}$, one defines an involutive automorphism $\sigma_p : G \longrightarrow G$ given by

$$\sigma_p(g) = s_p \circ g \circ s_p.$$

If F is the closed subgroup of G consisting of the elements fixed by σ_p, then it is easy to show that $F_0 \subseteq K \subseteq F$, where F_0 denotes the identity component of F.

Now, we identify the Lie algebra $\mathbf{g}$ with T_eG where e is the identity. Then we obtain an involutive Lie algebra automorphism $\theta_p = d\sigma_p : \mathbf{g} \longrightarrow \mathbf{g}$ charaterized by

$$\theta_p(X) = \frac{d}{dt}\sigma_p(e^{tX})|_{t=0}$$

or

$$\sigma_p(e^{tX}) = e^{t(\theta_p(X))}$$

for all $X \in \mathbf{g}$ and $t \in \mathbb{R}$, where e^{tX} denotes the 1-parameter subgroup of G determined by X.

Since $\theta_p^2 = I$, we may write,

$$\mathbf{g} = \mathbf{h} \oplus \mathbf{m}$$

where $\mathbf{h} = \{X \in \mathbf{g} : \theta_p(X) = X\}$ and $\mathbf{m} = \{X \in \mathbf{g} : \theta(X) = -X\}$.

Lemma 2.1.7 $\theta_p[X, Y] = [\theta_p X, \theta_p Y]$ *for all* $X, Y \in \mathbf{g}$. *Thus,* θ_p *is a Lie algebra homomorphism.*

Theorem 2.1.8 $[\mathbf{h}, \mathbf{h}] \subseteq \mathbf{h}, [\mathbf{h}, \mathbf{m}] \subseteq \mathbf{m}, [\mathbf{m}, \mathbf{m}] \subseteq \mathbf{h}$.

Proof : Since $\theta_p^2 = I, \theta_p$ has eigenvalues -1 and 1. Therefore, $\mathbf{h}$ is the eigenspace with eigenvalue 1 and $\mathbf{m}$ is the eigenspace with eigenvalue -1. If X is an eigenvector with eigenvalue λ, Y one with eigenvalue μ, then, since θ_p preserves brackets, $[X, Y]$ is an eigenvector with eigenvalue $\lambda\mu$. The result follows. □

Since $\mathbf{h}$ is a subspace of $\mathbf{g}$ and closed with respect to the Lie bracket, $\mathbf{h}$ is a subalgebra and ,in fact, $\mathbf{h}$ is the Lie algebra corresponding to $G_p = \{g \in G : gp = p\}$.

Definition 2.1.9 *For each $\phi \in G$, one defines the inner automorphism of G by conjugation:*

$$\begin{aligned} C_\phi \ &: \ G \longrightarrow G \\ &\quad g \longmapsto \phi g \phi^{-1} \end{aligned}$$

One denotes the group of vector space automorphisms of $\mathbf{g}$ by $Gl(\mathbf{g})$.

Definition 2.1.10 *For each $\phi \in G$, one defines a Lie algebra automorphism $Ad(\phi) : \mathbf{g} \longrightarrow \mathbf{g}$ by*

$$Ad(\phi) = dC_\phi : T_eG \longrightarrow T_eG$$

where $e \in G$ is the identity element.

The map $Ad : G \longrightarrow Gl(\mathbf{g})$ is called the adjoint representation of G.

Definition 2.1.11 *For each $X \in \mathbf{g}$, one defines a linear transformation $adX : \mathbf{g} \longrightarrow \mathbf{g}$ by*

$$(adX)(Y) = [X, Y]$$

where [,] is the Lie bracket of vector fields on G

The point $p \in \tilde{M}$ induces a map $p : G \longrightarrow \tilde{M}$ given by $p(g) = g(p)$. Identifying $\mathbf{g}$ with T_eG, we obtain a homomorphism

$$dp : \mathbf{g} \longrightarrow T_p\tilde{M}$$

whose kernel is precisely $\mathbf{h}$. The restriction $dp : \mathbf{m} \longrightarrow T_p\tilde{M}$, given by $X \mapsto X(p)$ for $X \in \mathbf{m}$, is an isomorphism. One defines an inner product Q on $\mathbf{m}$ by

$$Q(X,Y) =< dp(X), dp(Y) >_p$$

for all $X, Y \in \mathbf{m}$, where $<,>_p$ denotes the inner product on $T_p\tilde{M}$. From the definition of G_p and theorem 2.1.8, it is easy to see that $Ad(G_p)$ leaves $\mathbf{m}$ invariant and

$$Q(Ad(\phi)X, Ad(\phi)Y) = Q(X,Y)$$

for all $\phi \in G_p$ and for all $X, Y \in \mathbf{m}$.

2.1.5 Curvature tensor of $\tilde{M}$

Identifying $T_p\tilde{M}$ with $\mathbf{m}$ by means of the isomorphism $dp : \mathbf{m} \longrightarrow T_p\tilde{M}$, the skew symmetric curvature tensor of $\tilde{M}$ at p is given by

$$R(X,Y)Z = -ad[X,Y](Z) = -[[X,Y],Z]$$

for all $X, Y, Z \in \mathbf{m}$.

In particular, if $[X,Y] = 0$, then $R(X,Y) \equiv 0$. For the proof, which is a standard computation, we refer the reader to [W, p.245]

2.1.6 Killing form and classification of symmetric spaces

Definition 2.1.12 *The Killing form of* $\mathbf{g}$ *is the bilinear form* $B : \mathbf{g} \times \mathbf{g} \longrightarrow \mathbb{R}$ *given by*

$$B(X,Y) = tr(adX \circ adY)$$

Lemma 2.1.13 *The Killing form* B *of* $\mathbf{g}$ *is symmetric.* B *is invariant under automorphisms of* $\mathbf{g}$ *(i.e.* $B(\sigma X, \sigma Y) = B(X,Y)$*, for all* $\sigma \in \mathbf{g}$*). In particular,*

$$B((Ad\phi)X, (Ad\phi)Y) = B(X,Y) \qquad \text{for all } X, Y \in \mathbf{g}, \phi \in G.$$

Also,

$$B((adX)Y, Z) + B(Y, (adX)Z) = 0 \qquad \text{for all } X, Y, Z \in \mathbf{g}.$$

Given a point p in a symmetric space $\tilde{M}$, let $G = I_0(\tilde{M})$ and let $\mathbf{g} = \mathbf{h} + \mathbf{m}$ be the decomposition of $\mathbf{g}$ into eigenspaces of the involution $\theta_p : \mathbf{g} \longrightarrow \mathbf{g}$. Let B_m denote the restriction of the Killing form B to $\mathbf{m} \times \mathbf{m}$

Definition 2.1.14 $\tilde{M}$ *is said to be of compact (noncompact, Euclidean) type if* B_m *is negative definite (positive definite, identically zero).*

2.1.7 Holonomy of $\tilde{M}$ at p

Proposition 2.1.15 *Let* $\tilde{M}$ *be a symmetric space of noncompact type. Let* p *be a point of* $\tilde{M}$*, and let* $K = \{g \in G : g(p) = p\}$*. Then the holonomy group* Φ_p *at* p *equals* $dK = \{d\phi : \phi \in K\}$

A detailed discussion of this result can be found in [H, p.162].

2.1.8 Rank of a symmetric space $\tilde{M}$

Definition 2.1.16 *Let $\tilde{M}$ be any simply connected manifold. If $k \geq 1$ is an integer, then an k-flat in $\tilde{M}$ is a complete, totally geodesic submanifold of $\tilde{M}$ isometric to the flat Euclidean space $\mathbb{R}^k$.*

Definition 2.1.17 *The rank of a symmetric space $\tilde{M}$ is the dimension of the largest k-flat in $\tilde{M}$.*

In fact, if $v \in S\tilde{M}$ is arbitrary, then the geodesic γ such that $\gamma'(0) = v$ is contained in a k-flat of $\tilde{M}$.

Let $\mathbf{a}$ be an abelian subspace of $\mathbf{m}$. One puts

$$A = exp\ \mathbf{a}$$

where exp is the exponential map $\mathbf{g} \longrightarrow G$.
A then is a Lie subgroup of G.
For $g_1, g_2 \in A$, we have

$$g_1 g_2 = g_2 g_1$$

because for any two commuting elements $X, Y \in \mathbf{g}$,

$$e^{X+Y} = e^X e^Y = e^Y e^X.$$

Thus, A is an abelian Lie group.
On the other hand, because of $\mathbf{a} \subset \mathbf{m}$, A also is a subspace of $\tilde{M}$.

In fact, one shows that A is totally geodesic in $\tilde{M}$ and flat.
Therefore, the k-flat in a symmetric space $\tilde{M}$ containing a point $p \in \tilde{M}$ can be described algebraically as orbits $A(p)$, where $A = exp\ \mathbf{a}$ and $\dim(\mathbf{a}) = k$.

2.1.9 Regular and singular points at infinity

Definition 2.1.18 *A geodesic γ in $\tilde{M}$ is called regular if it is contained in one k-flat only; otherwise it is called singular. Tangent vectors of regular(singular) geodesic are called regular(singular).*

Remark. For each point $x \in \tilde{M}(\infty)$, let $G_x = \{g \in G = I_0(\tilde{M}) : g(x) = x\}$. Then one may show that G_x acts transitively on $\tilde{M}$. Therefore, a geodesic σ of $\tilde{M}$ asymptotic to a regular(singular) geodesic γ is also regular(singular). Hence, one defines a point $x \in \tilde{M}(\infty)$ be regular(singular) if the representation γ of x is regular(singular). The set of regular points at infinity, denoted $R(\infty)$, is a dense open subset of $\tilde{M}(\infty)$ that is invariant under $I(\tilde{M})$. If $\tilde{M}$ has rank $k \geq 2$, then the set of singular points at infinity is a closed, nowhere dense subset of $\tilde{M}(\infty)$ that is invariant under $I(\tilde{M})$.

2.2 The Symmetric Space $\tilde{M}_n = SL(n,\mathbb{R})/SO(n,\mathbb{R})$

For each integer $n \geq 2$, let

$$\begin{aligned}
\tilde{M}_n &= \{A \in GL(n,\mathbb{R}) : A^t = A, A \; positive \; definite, \; detA = 1\} \\
SL(n,\mathbb{R}) &= \{A \in GL(n,\mathbb{R}) : detA = 1\} \\
SO(n,\mathbb{R}) &= \{A \in Gl(n,\mathbb{R}) : A^t = A^{-1}, detA = 1\}
\end{aligned}$$

The group $G = SL(n,\mathbb{R})$ acts transitively on $\tilde{M}_n$ as follows: given $g \in SL(n,\mathbb{R})$ and $A \in \tilde{M}_n$, let $g(A) = gAg^t$. The subgroup of $SL(n,\mathbb{R})$ that fixes the identity $I \in \tilde{M}_n$ is precisely $K = SO(n,\mathbb{R})$, a maximal compact subgroup of G. It is easy to show that $SL(n,\mathbb{R}) = I_0(\tilde{M}_n)$. We therefore have the representation

$$\tilde{M}_n = SL(n,\mathbb{R})/SO(n,\mathbb{R}).$$

Let $\wp = \{\ n \times n$ symmetric matrices with trace zero $\}$. Then the matrix exponential map $exp : \wp \longrightarrow \tilde{M}_n$ given by

$$exp(A) = e^A = I + \sum_{k=1}^{\infty} \frac{A^k}{k!}$$

is a bijection.

The exponential map exp also yields a diffeomorphism between $\tilde{M}_n$ and a Euclidean space of dimension $\frac{1}{2}n(n+1)$.

Therefore, one uses the exponential map to identify $\wp$ with the tangent space $T_I\tilde{M}_n$ (i.e. $X \in \wp$ is associated with the initial velocity of $t \longrightarrow exp(tX)$.).

2.2.1 Metric on $T_I\tilde{M}_n$

Identifying $T_I\tilde{M}_n$ with $\wp$, one defines an inner product $<,>$ on $\wp$ given by

$$< X, Y >_I = Trace(XY^t) = Trace(XY).$$

This is the restriction of the Euclidean inner product on $\mathbb{R}^{n^2}$.

Since $SL(n,\mathbb{R})$ acts transitively on $\tilde{M}_n$, one can define an inner product on $T_P\tilde{M}_n$ as follows: if $P \in \tilde{M}_n$ and if $g \in SL(n,\mathbb{R})$ such that $g(I) = P$, then an inner product on $T_I\tilde{M}_n$ is given by

$$< dg(X), dg(Y) >_P = < X, Y >_I \qquad for\ all\ X, Y \in \wp.$$

One may show that the inner product on $T_I\tilde{M}_n$ is $SO(n,\mathbb{R})$-invariant. Although g is not uniquely determined in the definition of $<,>_P$, if $g_1, g_2 \in SL(n,\mathbb{R})$ such that $g_1(I) = g_2(I) = P$, then there exists $k \in SO(n,\mathbb{R})$ such that $g_1 = g_2 k$.

Therefore,

$$< dg_1(X), dg_1(Y) >_P = < dg_2 k(X), dg_2 k(Y) >_P = < dg_2(X), dg_2(Y) >_P$$

Hence, the inner product is well-defined at P.

2.2.2 Geodesic and symmetries of $\tilde{M}_n$

A unit speed geodesic of $\tilde{M}_n$ that start at the identity I can be described as follows: for $X \in \wp$ and $\|X\|^2 = \text{Trace}(X^2) = 1$, let

$$\gamma_X(t) = exp(tX)(I).$$

Since $SL(n, \mathbb{R}) = I_0(\tilde{M}_n)$ acts transitively on $\tilde{M}_n$, a geodesic passing through a point $p \in \tilde{M}_n$ can be obtained by acting some $g \in SL(n, \mathbb{R})$ on a geodesic at I.

The map $X \longrightarrow X^{-1}$ for $X \in \tilde{M}_n$ defines the geodesic symmetry S_I at $I \in \tilde{M}_n$. If $P \in \tilde{M}_n$ such that $g(I) = P$ for some $g \in SL(n, \mathbb{R})$, then the geodesic symmetry at P is given by

$$S_P = g \circ S_I \circ g^{-1}.$$

Moreover, if $g(I) = P, Q \in \tilde{M}_n$, let $R = g^{-1}(Q)$, then

$$S_P(Q) = g \circ S_I \circ g^{-1}(Q) = gR^{-1}g^t = gg^tQ^{-1}gg^t = PQ^{-1}P.$$

2.2.3 Curvature of $\tilde{M}_n$

With the identification $\wp = T_I\tilde{M}_n$, the curvature tensor of $\tilde{M}_n$ is given by

$$R(X,Y)Z = -[[X,Y],Z]$$

for all $X, Y \in \wp$, where $[\ ,\ ]$ is the Lie bracket.

2.2.4 Rank and flats in $\tilde{M}_n$

From the above discussion and the definition of Killing form. One shows that

$$\tilde{M}_n = SL(n, \mathbb{R})/SO(n, \mathbb{R})$$

is a symmetric space of noncompact type with nonpositive sectional curvature. Moreover, one puts

$$\mathbf{g} = sl(n, \mathbb{R}), \mathbf{k} = so(n, \mathbb{R}), \mathbf{m} = \{X \in sl(n, \mathbb{R}) : X^t = X\}$$

where g and k are Lie algebra of $SL(n, \mathbb{R})$ and $SO(n, \mathbb{R})$ respectively. Then because of

$$X = \frac{1}{2}(X - X^t) + \frac{1}{2}(X + X^t)$$

for all $X \in sl(n, \mathbb{R})$, one obtains

$$\mathbf{g} = \mathbf{k} \oplus \mathbf{m}.$$

By using exp, one also shows that the maximal k-flat of $\tilde{M}_n$ containing the identity I bijectively correspond to the maximal abelian subspace of $\mathbf{m}$. Since $\wp$ is the maximal abelian subspace of $\mathbf{m}$ and $\dim(\wp) = n - 1$. It follows that $rank(\tilde{M}_n) = n - 1$. See [J, chapter 6] for details.

2.2.5 Holonomy of $\tilde{M}_n$ at I

For any element $g \in SO(n, \mathbb{R})$, the maximal compact subgroup of $SL(n, \mathbb{R})$, one obtains

$$g(exp(X)) = g(exp(X))g^t = g(exp(X))g^{-1} = exp(gXg^{-1})$$

for all $X \in \wp$.

By using proposition 2.1.15, one has the following condition.

Proposition 2.2.1 *Two unit vectors $X, Y \in \wp$ lie in the same orbit of the holonomy group Φ_I if and only if $Y = gXg^{-1}$ for some $g \in SO(n, \mathbb{R})$. Therefore the orbits of the holonomy group are in one-one correspondence with the n-tuples $(\lambda_1, \ldots \lambda_n)$ such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, $\sum_{i=1}^{n} \lambda_i = 0$ and $\sum_{i=1}^{n} {\lambda_i}^2 = 1$, where the $\{\lambda_i\}$ are the eigenvalues of X in $\wp$.*

Given a unit vector $X \in \wp$, then one may show that $\gamma_X(t)$ is regular if and only if the eigenvalues of X are all distinct.

2.2.6 Eigenvalue-flag pair for a point in $\tilde{M}_n(\infty)$

Definition 2.2.2 *A set of nonzero vector subspaces $F = (V_1, \ldots, V_k)$ of $\mathbb{R}^n$ is said to be a flag if $V_k = \mathbb{R}^n$ and V_i is a proper subspace of V_{i+1} for every i.*

Definition 2.2.3 *A flag $F = (V_1, \ldots, V_k)$ in $\mathbb{R}^n$ is regular if $k = n$ or equivalently if $m_i = \dim V_i - \dim V_{i-1} = 1$ for all i.*

Let

$$\wp_1 = \{\text{symmetric } n \times n \text{ matrices } X \text{ with } \mathrm{Trace}(X) = 0 \text{ and } \mathrm{Trace}(X^2){=}1\}.$$

Then, by proposition 1.2.13, $\tilde{M}_n(\infty)$ can be identified with the unit vectors in $T_I\tilde{M}_n$, which can in turn be identified with $\wp_1$. In particular, $x \in \tilde{M}_n(\infty)$ is associated with element $X \in \wp_1$ such that $\gamma_X(\infty) = x$, where $\gamma_X(t) = exp(tX)(I)$.

Let $\{\lambda_1(x), \dots, \lambda_k(x)\}$ be the distinct eigenvalues of X, arranged so that $\lambda_1(x) \geq \lambda_2(x) \geq \dots \geq \lambda_k(x)$. Now let $E_i(x)$ be the eigenspace of X associated to $\lambda_i(x)$ and let

$$V_i(x) = E_1(x) \oplus \cdots \oplus E_i(x) \qquad \text{for } 1 \leq i \leq k.$$

Then one obtains a flag of subspace $F(x) = (V_1(x), \dots, V_k(x))$ such that $V_1(x) \subseteq V_2(x) \subseteq \cdots \subseteq V_k(x) = \mathbb{R}^n$. Therefore, for each $x \in \tilde{M}_n(\infty)$, one associates a flag $F(x) = (V_1(x), \dots, V_k(x))$ in $\mathbb{R}^n$ and a vector $\lambda(x) = (\lambda_1(x), \dots, \lambda_k(x))$ such that

a, $\lambda_1(x) \geq \lambda_2(x) \geq \cdots \geq \lambda_k(x)$,

b, $\sum\limits_{i=1}^{k} m_i\lambda_i(x) = 0$ where $m_i = \dim V_i(x) - \dim V_{i-1}(x)$,

c, $\sum\limits_{i=1}^{k} m_i{\lambda_I}^2(x) = 1$.

Conditions (b) and (c) are equivalent to the conditions that $X \in \wp$ have $\text{Trace}(X) = 0$ and $\|X^2\| = 1$. Conversely, for any flag of subspaces $F = (V_1, \dots, V_k)$ and vector $\lambda = (\lambda_1, \dots, \lambda_k)$ that satisfy conditions (a), (b) and (c) for some k with $2 \leq k \leq n$, then there exists a unique point $x \in \tilde{M}_n(\infty)$ such that $\lambda(x) = \lambda$ and $F(x) = F$.

Hence, the set of points at infinity $\tilde{M}_n(\infty)$ can be identified with the set of eigenvalue-flag pairs that satisfy (a), (b) and (c).

2.2.7 Action of $I_0(\tilde{M}_n)$ on $\tilde{M}_n(\infty)$

If $F = (V_1, \dots, V_k)$ is a flag in $\mathbb{R}^n$ and an element $g \in SL(n, \mathbb{R})$, one defines the action of $SL(n, \mathbb{R})$ on $F = (V_1, \dots, V_k)$ as

$$g(F) = (g(V_1), \dots, g(V_k)).$$

Proposition 2.2.4 *Let $g \in SL(n, \mathbb{R})$ and $x \in \tilde{M}_n(\infty)$ be given, and let $(\lambda(x), F(x))$ be the eigenvalue-flag pair associated to x. Then $(\lambda(x), F(x))$ is the eigenvalue-flag pair associated to $g(x)$, or equivalently*

a, $\lambda(g(x)) = \lambda(x)$,

b, $F(g(x)) = g(F(x))$.

Sketch of the proof. One considers only the case that x is a regular point at infinity. Since $SL(n, \mathbb{R})$ acts transitively on the space of regular flags, one may reduce to the case that

$$F(x) = (V_1, V_2, \dots, V_k)$$

where $V_i = Span\{e_1, e_2, \dots, e_i\}$ and $\{e_1, e_2, \dots, e_n\}$ is the standard basis of $\mathbb{R}^n$.

In this case, one shows that $g(x) = x$ for $g \in SL(n, \mathbb{R})$ if and only if g preserves the flag $F(x)$ if and only if g is an upper triangular $n \times n$ matrix. On the other hand, for each $\phi \in SO(n, \mathbb{R})$ and $x = \gamma_X(\infty)$ for $X \in \wp$ with $\|X^2\| = 1$, one has

$$(\phi \circ \gamma_X)(s) = \phi \gamma_X(s) \phi^t = \phi(exp(tX)(I))\phi^{-1} = exp(t\phi X \phi^{-1})(I) = \gamma_{\phi X \phi^{-1}}(s),$$

therefore, $\phi(x) = (\phi \circ \gamma_X)(\infty) = \gamma_{\phi X \phi^{-1}}(\infty)$.

Then assertions (a) and (b) of the proposition are clear for $\phi \in SO(n, \mathbb{R})$ and

x as above. Since any element $h \in SL(n, \mathbb{R})$ can be written as $h = \phi g$, where $g \in SL(n, \mathbb{R})$ is upper triangular and $\phi \in SO(n, \mathbb{R})$, then assertions (a) and (b) hold for all elements of $SL(n, \mathbb{R})$.

2.2.8 Flags in opposition

Definition 2.2.5 *Two flags $F = (V_1, \dots, V_k)$ and $F' = (V_1', V_2', \dots, V_k')$ are said to be in opposition if $k = r$ and $V_i \oplus V_{k-i}' = \mathbb{R}^n$ for every i.*

A flag $F = (V_1, \dots, V_k)$ determines an inverse flag $F^{-1} = (V_1^*, V_2^*, \dots V_k^*)$, where V_i^* is the orthogonal complement in $\mathbb{R}^n$ of V_{k-i} for each i. Clearly, F and F^{-1} are in opposition.

Fact. Two flags F_1 and F_2 are in opposition if and only if there exists an element $g \in SL(n, \mathbb{R})$ such that $g(F_1) = F_1$ and $g(F_1^{-1}) = F_2$.

2.2.9 Joining points at infinity

One gives a precise condition in terms of eigenvalue-flag pairs for those points $x, y \in \tilde{M}_n(\infty)$ that can be joined by a geodesic of $\tilde{M}_n$.

Proposition 2.2.6 *Let x, y be distinct points of $\tilde{M}_n(\infty)$, let $(\lambda(x), F(x))$ and $(\lambda(y), F(y))$ be the corresponding eigenvalue-flag pairs. Then there exists a geodesic γ of $\tilde{M}_n$ such that $\gamma(\infty) = x$ and $\gamma(-\infty) = y$ if and only if*

1. *$F(x)$ and $F(y)$ are in opposition,*

2. *$\lambda_i(x) = -\lambda_{k-i+1}(y)$ for $1 \leq i \leq k$ where $\lambda(x) = (\lambda_1(x), \lambda_2(x), \dots, \lambda_k(x))$ and $\lambda(y) = (\lambda_1(y), \lambda_2(y), \dots, \lambda_k(y))$.*

Before the proof of this proposition, we first prove a useful lemma. It states that:

Lemma 2.2.7 *Let $\tilde{M}$ be a symmetric space of noncompact type and rank $k \geq 2$. Let x be any point of $\tilde{M}(\infty)$, and let y be any point of $\tilde{M}(\infty)$ that can be joined to x by a geodesic of $\tilde{M}$. Let $G = I_0(\tilde{M})$. Then:*
The set of points in $\tilde{M}(\infty)$ to which x can be joned by a geodesic of $\tilde{M}$ is the orbit $G_x(y)$, where $G_x = \{g \in G : gx = x\}$.

Proof : Let $\gamma(t)$ be a unit speed geodesic of $\tilde{M}$ such that $\gamma(\infty) = x$ and $\gamma(-\infty) = y$. If $z = \phi(y)$ for some $\phi \in G_x$ and if $\sigma = \phi \circ \gamma$, then $\sigma(\infty) = x$ and $\sigma(-\infty) = z$.
Hence any point of $G_x(y)$ can be joined to x by a geodesic of $\tilde{M}$.
Conversely, let $z \in \tilde{M}(\infty)$ be a point such that $\sigma(\infty) = x$ and $\sigma(-\infty) = z$ for some geodesic σ of $\tilde{M}$. Let $p = \gamma(0)$ where γ is a geodesic of $\tilde{M}$ with $\gamma(\infty) = x$ and $\gamma(-\infty) = y$. One uses the fact that G_x acts transitively on $\tilde{M}$. Then there exists an element $\phi \in G_x$ such that $\phi(p) = \sigma(0)$. One concludes that $\phi \circ \gamma = \sigma$ since $(\phi \circ \gamma)(0) = \sigma(0)$ and $(\phi \circ \gamma)(\infty) = \sigma(\infty) = x$. Therefore,

$$z = \sigma(-\infty) = (\phi \circ \gamma)(-\infty) = \phi(y) \in G_x(y),$$

which completes the proof of the lemma. $\square$

Proof of proposition 2.2.6. With the identification of

$$\tilde{M}_n = SL(n, \mathbb{R})/SO(n, \mathbb{R}),$$

let $p = I \cdot SO(n, \mathbb{R})$. and let X be the unit vector in $\wp$ such that $\gamma_X(\infty) = x$. If $z = \gamma_{px}(-\infty)$, then $z = \gamma_{(-X)}(\infty)$ and it is easy to show that $\lambda_i(x) =$

$-\lambda_{k-i+1}(z)$ for $1 \leq i \leq k$.

If there exists a geodesic γ of $\tilde{M}_n$ with $\gamma(\infty) = x$ and $\gamma(-\infty) = y$, then by the lemma above, there exists an element $g \in SL(n, \mathbb{R})$ such that $g(x) = x$ and $g(z) = y$. Hence $\lambda(y) = \lambda(g(z)) = \lambda(z)$ and $F(y) = F(g(z)) = g(F(z))$ is in opposition to $g(F(x)) = F(g(x)) = F(x)$ since $F(z)$ is in opposition to $F(x)$.

Conversely, suppose that (1) and (2) hold for points $x, y \in \tilde{M}(\infty)$. By (1) and the fact above there exists an element $g \in SL(n, \mathbb{R})$ such that

$$g(F(x)) = F(x) \text{ and } g(F(z)) = F(y).$$

We note that $g(x) = x$ by the proof of proposition 2.2.4 since g leaves the flag $F(x)$ invariant. Hence $x = g(x)$ can be joined to $g(z)$ by the geodesic $g \circ \gamma_X$. One observes that

$$F(y) = g(F(z)) = F(g(z)) \text{ and } \lambda(y) = \lambda(z) = \lambda(g(z))$$

by (2). Therefore, $y = g(z)$ since $g(z)$ have the same eigenvalue-flag pairs.

Chapter 3

Group Action

3.1 Action of Isometries on $\tilde{M}(\infty)$

3.1.1 Fundamental group as a group of isometries

A simply-connected manifold $\tilde{M}$ is diffeomorphic to a Euclidean space, and it follows from the theory of covering spaces that if M is any complete manifold of nonpositive sectional curvature, then the homotopy groups $\pi_k(M)$ vanish for all $k \geq 2$.

Since the topology of M is contained in the fundamental group $\pi_1(M)$ in a certain sense, therefore, it is reasonable to look for relations between the geometric properties of a complete manifold M of nonpositive sectional curvature and the algebraic structure of $\pi_1(M)$.

The most effective way to study $\pi_1(M)$ is to regard $\pi_1(M)$ as a freely, properly discontinuous group of isometries Γ of the universal cover $\tilde{M}$ and Γ is a discrete group of isometries of $\tilde{M}$ that contains no elliptic elements. Then the theory of covering spaces says that M can be expressed as such a quotient space $\tilde{M}/\Gamma$.

One can use information about the action of Γ on $\tilde{M}$ or on $\tilde{M}(\infty)$ to obtain information about $\pi_1(M)$.

Definition 3.1.1 *An isometry ϕ of $\tilde{M}$ is called elliptic if ϕ fixes some point of $\tilde{M}$.*

Definition 3.1.2 *A group of isometries $\Gamma \subseteq I(\tilde{M})$ is discrete if for every compact subset A of $\tilde{M}$, there exist at most finitely many isometries ϕ of Γ such that $\phi(A) \cap A$ is nonempty.*

Proposition 3.1.3 *Let M be a complete manifold with nonpositive sectional curvature. Then every element of $\pi_1(M)$ has infinite order, except for the identity.*

Proof : By theory of covering spaces, M can be expressed as $\tilde{M}/\Gamma$ where $\tilde{M}$ is simply-connected and $\Gamma \subseteq I(\tilde{M})$ is discrete and has no elliptic elements. Identifying $\pi_1(M)$ with a group Γ of isomtries of $\tilde{M}$ where $\tilde{M}$ is the universal covering of M.

Suppose there exists $\phi \neq 1$ in Γ with finite order. Let Γ^* be the finite cyclic subgroup of Γ generated by ϕ. By the Cartan fixed point theorem, the group Γ^* fixes some point p of $\tilde{M}$, but this contradicts the fact that Γ has no elliptic elements except the identity. $\square$

3.1.2 Lattices

Definition 3.1.4 *If $\Gamma \subseteq I(\tilde{M})$ is a discrete group, then*

$$vol(\Gamma) = \sup\{vol(O) \ : \ O \text{ is a open subset of } \tilde{M} \text{ such that } \phi(O) \text{ is disjoint from } O \text{ for all } \phi \in \Gamma \text{ with } \phi \neq 1\}.$$

Definition 3.1.5 *A discrete group $\Gamma \subseteq I(\tilde{M})$ is a lattice if $vol(\Gamma)$ is finite.*

Definition 3.1.6 *A lattice $\Gamma \subseteq I(\tilde{M})$ is called uniform if the quotient space $\tilde{M}/\Gamma$ is compact and otherwise is called nonuniform.*

Examples

1. Quotients of Euclidean spaces.

   If $v_1, \ldots, v_n$ are any linearly independent vectors in $\mathbb{R}^n$, then let Γ be the group of translations of $\mathbb{R}^n$ of the form $\sum_{i=1}^{n} m_i v_i$, where $\{m_i\}$ are integers. The quotient manifold $\mathbb{R}^n/\Gamma$ is an n-torus.

   See [W, p.98-136] for a discussion of flat manifolds.

2. Quotients of hyperbolic spaces.

   If M is any compact orientable surface of genus $g \geq 2$, then there exists a uniform lattice Γ in $PSL(2, \mathbb{R})$, the connected isometry group of the hyperbolic plane, such that M is diffeomorphic to H^2/Γ. In particular, M admits a complete Riemannian metric with $K \equiv -1$. The set of such lattices Γ is called Teichmüller space for M and it is diffeomorphic to a Euclidean space of dimension $6g - 6$.

3. The group $SL(2, \mathbb{Z})$ is a nonuniform lattice in $PSL(2, \mathbb{R}) = I_0(H^2)$.

4. By result of A. Borel [Bo], if $\tilde{M}$ is a symmetric space of noncompact type, then $I_0(\tilde{M})$, the connected isometry group of $\tilde{M}$, admits both uniform and nonuniform lattices.

3.1.3 Duality condition

Definition 3.1.7 *Points x, y in $\tilde{M}(\infty)$ are Γ-dual relative to a subgroup Γ of $I(\tilde{M})$ if there exists a sequence $\{\phi_n\} \subseteq \Gamma$ such that $\phi_n(p) \to x$ and ${\phi_n}^{-1}(p) \to y$ for every $p \in \tilde{M}$.*

Definition 3.1.8 *A subgroup $\Gamma \subseteq I(\tilde{M})$, not necessarily discrete, will be said to satisfy the duality condition if $\gamma(\infty)$ and $\gamma(-\infty)$ are Γ-dual for any geodesic γ of $\tilde{M}$.*

Proposition 3.1.9 *Let $\Gamma \subseteq I(\tilde{M})$ be any group, and let x, y be points in $\tilde{M}(\infty)$ that are Γ-dual. If $z \in \tilde{M}(\infty)$ is any point that can be joined to x, then $y \in \overline{\Gamma(z)}$.*

Proof : Let γ, σ be geodesics of $\tilde{M}$ such that $\gamma(\infty) = z, \sigma(\infty) = y$ and $\gamma(-\infty) = \sigma(-\infty) = x$.
Since Γ satisfies the duality condition, there exists a sequence $\{\phi_n\} \subseteq \Gamma$ such that

$$\phi_n p \to y \text{ and } {\phi_n}^{-1} p \to x \qquad\qquad \text{for all } p \in \tilde{M}.$$

By choosing $p \in \gamma$, we have,

$$\angle_p(\phi_n p, \phi_n z) = \angle_{{\phi_n}^{-1}p}(p, z) \leq \angle_p({\phi_n}^{-1} p, x) \to 0 \text{ as } n \to \infty$$

since

$$\angle_p({\phi_n}^{-1}p, z) + \angle_{\phi_n{-1}p}(p, z) \leq \pi.$$

Therefore, $$\phi_n z \to y \text{ as } n \to \infty$$ $\square$

Remark. For a fixed point $p \in \tilde{M}$, the duality of two points x and y is equivalent to the following: if U, V are arbitrary neighborhoods in $\tilde{M}$ of x, y respectively, then there exists $\phi \in \Gamma$ such that $\phi(p) \in U$ and $\phi^{-1}(p) \in V$. The set of points dual to x is closed in $\tilde{M}(\infty)$ and invariant under Γ.

Example Let $\Gamma = I_0(\tilde{M})$, where $\tilde{M}$ is a symmetric space. Then Γ satisfies the duality condition by the discussion of transvections in section 2.1.2

Proposition 3.1.10 *Let $\Gamma \subseteq I(\tilde{M})$ satisfy the duality condition. Then for any two points x, y in $\tilde{M}(\infty)$, the closed sets $\overline{\Gamma(x)}$ and $\overline{\Gamma(y)}$ are either disjoint or identical.*

This result is lemma 2.9 of [BBE].

Proposition 3.1.11 *Let $\tilde{M}$ have sectional curvature satisfying $K \leq c < 0$, and let $\Gamma \subseteq I(\tilde{M})$ be a group that satisfies the duality condition. Then $\overline{\Gamma(x)} = \tilde{M}(\infty)$ for every point $x \in \tilde{M}(\infty)$.*

Proof : For two points x, y in $\tilde{M}(\infty)$, let $z \in \tilde{M}(\infty)$ be distinct from both x and y.
Since $K \leq c < 0$, by discussion in section 1.2.5, then there exist geodesics γ_1 and γ_2 of $\tilde{M}$ that join z to x and z to y respectively. Since Γ satisfies the

duality condition, z is Γ-dual to both x and y.

Therefore, it follows from proposition 3.1.9, that $y \in \overline{\Gamma(x)}$. $\square$

3.1.4 Geodesic flows

We consider the geodesic flow $\{g^t\}$ on the unit tangent bundle SM of a complete manifold M of $K(M) \leq 0$, and relate the action of $\{g^t\}$ on SM to the action of the fundamental group Γ of M on $\tilde{M}(\infty)$, where $M = \tilde{M}/\Gamma$.

Proposition 3.1.12 *Let $M = \tilde{M}/\Gamma$ be a complete manifold of $K(M) \leq 0$ and finite volume, where $\tilde{M}$ is simply connected and Γ is a discrete group of isometries of $\tilde{M}$ without elliptic elements. Then the following properties are equivalent:*

1. *The geodesic flow $\{g^t\}$ has a dense orbit in the unit tangent bundle SM.*

2. *The group Γ has a dense orbit in $\tilde{M}(\infty)$.*

3. *Every orbit of Γ in $\tilde{M}(\infty)$ is dense in $\tilde{M}(\infty)$.*

This result is contained in theorem 4.14 of [E1] and the discussion in section 3.1.3

Corollary 3.1.13 *Let M be a complete manifold with finite volume and $K(M) \leq c < 0$. Then the geodesic flow has a dense orbit in the unit tangent bundle SM.*

Proof : Since M can be expressed as a quotient manifold $\tilde{M}/\Gamma$ where $\tilde{M}$ is simply connected and $\Gamma \subseteq I(\tilde{M})$ satisfies the duality condition, then by proposition 3.1.11, $\overline{\Gamma(x)} = \tilde{M}(\infty)$ for all $x \in \tilde{M}(\infty)$. Therefore, by proposition 3.1.12, the result follows. □

Corollary 3.1.14 *Let M be a compact manifold with $K \equiv 0$. Then the geodesic flow does not have a dense orbit in SM.*

Proof : Since $K \equiv 0$, one may write

$$M = \mathbb{R}^n/\Gamma$$

where $n = \dim(M)$ and Γ is a uniform lattice in $I(\mathbb{R}^n)$.
A theorem of Bieberbach says that Γ contains a normal free abelian subgroup Γ_0 with rank n and finite index in Γ, and Γ_0 consists of translations of $\mathbb{R}^n$. Since the translations in $\mathbb{R}^n$ fix every point of $\mathbb{R}^n(\infty)$, any orbit of Γ in $\mathbb{R}^n(\infty)$ is finite. Hence Γ has no dense orbit in $\mathbb{R}^n(\infty)$. Finally, the result follows proposition 3.1.12. □

Remark. If $\tilde{M}$ is any symmetric space of noncompact type and rank $k \geq 2$ and if M is any finite volume quotient manifold, then the geodesic flow does not have a dense orbit in the unit tangent bundle SM. Since the set A consisting of the singular points in $\tilde{M}(\infty)$ is a closed, nowhere dense subset of $\tilde{M}(\infty)$ that is invariant under $I(\tilde{M})$.

3.2 Action of Geodesic Symmetries on $\tilde{M}(\infty)$

For each point $p \in \tilde{M}$, if $s_p : \tilde{M} \longrightarrow \tilde{M}$ is the geodesic symmetry, then the diffeomorphism s_p extends to a homeomophism $s_p : \tilde{M}(\infty) \longrightarrow \tilde{M}(\infty)$ if one defines

$$s_p(x) = \gamma_{px}(-\infty)$$

for all $x \in \tilde{M}(\infty)$.

Therefore, the symmetry diffeomorphism group G^* acts on $\tilde{M}(\infty)$ by homeomorphisms.

Let Φ_p be the holonomy group at p. If $q \in \tilde{M}$ is distinct from p and σ is a geodesic joining q to p, then

$$\Phi_q = (P_\sigma^{-1}) \circ \Phi_p \circ P_\sigma$$

where P_σ is a parallel translation operator along σ. The following result states that the action of G^* on $\tilde{M}(\infty)$ is related to the action of Φ_p acting on the unit vectors of $T_p\tilde{M}$ for every point $p \in \tilde{M}$.

A detailed discussion and proof of the following result can be found in [E3].

Theorem 3.2.1 *Let $p \in \tilde{M}$ and $x, y \in \tilde{M}(\infty)$ be arbitrarily given points. If $V(p, y)$ lies in the holonomy orbit $\Phi_p(V(p, x))$, then $y \in \overline{G^*(x)}$.*

For the proof of theorem 3.2.4 below, we need the following result of M. Berger[Be].

Theorem 3.2.2 *Let M be an irreducible complete simply-connected Riemannian manifold, and let p be any point in M. If the holonomy group Φ_p has a proper closed invariant subset in the sphere of unit vectors in T_pM, then M is isometric to a symmetric space with rank $k \geq 2$.*

Definition 3.2.3 *A Riemannian manifold is called irreducible if no one of its finite coverings is a Riemannian product.*

By theorem 3.2.1, the holonomy group Φ_p has a proper, closed invariant subset in the sphere of unit vectors in $T_p\tilde{M}$ for any point $p \in \tilde{M}$ if G^* admits a proper closed invariant subset in $\tilde{M}(\infty)$.

Hence, by the above two theorems, one can conclude the following result.

Theorem 3.2.4 *Let $\tilde{M}$ be irreducible and suppose that G^* admits a proper closed invariant subset in $\tilde{M}(\infty)$. Then $\tilde{M}$ is isometric to a symmetric space of noncompact type and rank $k \geq 2$.*

Definition 3.2.5 *A subset $X \subseteq \tilde{M}(\infty)$ is called involutive if X is invariant under the symmetry diffeomorphism group G^*.*

Fact. If $\Gamma \subseteq I(\tilde{M})$ satisfies the duality condition, then finding proper, closed G^*-invariant subsets in $\tilde{M}(\infty)$ is closely related to finding proper, closed Γ-invariant subsets in $\tilde{M}(\infty)$.

Proposition 3.2.6 *Let $\Gamma \subseteq I(\tilde{M})$ satisfy the duality condition, and let $X \subseteq \tilde{M}(\infty)$ be closed and Γ-invariant. Then $X \cup s_p(X)$ and $X \cap s_p(X)$ are closed involutive subsets of $\tilde{M}(\infty)$ for any point p in $\tilde{M}$.*

Proof : We first show that X is G_e^*-invariant where G_e^* consists of those elements of G^* that are products of an even number of geodesic symmetries. It suffices to show that $s_p s_q(x) \in \overline{\Gamma(x)}$ for any points p, q in $\tilde{M}$ and $x \in X$. If $y = s_q(x)$ and $z = s_p(x) = s_p s_q(x)$, then y is Γ-dual to z since Γ satisfies the duality condition.

By proposition 3.1.9, $s_p s_q(x) = z \in \overline{\Gamma(x)} \subseteq X$. Since G_e^* is normal in G^* and X is G^*-invariant then

$$s_p^{-1}(x) G_e^*(X) s_p(x) \subset G^*(X)$$

for some $p \in \tilde{M}$ and $x \in X$.

Therefore,

$$s_p^{-1}(x) G_e^*(X) s_p(x) \subset X.$$

Since $x \in X$ and $p \in \tilde{M}$ are arbitrary, we have

$$G_e^*(X) s_p(X) \subset s_p(X).$$

This implies that $s_p(X)$ is G_e^*-invariant.

Hence, $X \cap s_p(X)$ and $X \cup s_p(X)$ are both involutive. Moreover, if $X \cup s_p(X) = \tilde{M}(\infty)$, then $X \cap s_p(X)$ must be nonempty since $\tilde{M}(\infty)$ is connected which completes the proof. □

Theorem 3.2.7 *Let $\tilde{M}$ be an irreducible, complete, simply-connected manifold of nonpositive sectional curvature whose isometry group $I(\tilde{M})$ satisfies the duality condition. Then $\tilde{M}$ is isometric to a symmetric space of noncompact type and rank $k \geq 2$ if and only if $\tilde{M}(\infty)$ admits a proper closed subset invariant under $I(\tilde{M})$.*

Proof : If $\tilde{M}$ is a symmetric space of noncompact type and rank $k \geq 2$, then by remark in section 2.1.9, the set A of singular points in $\tilde{M}(\infty)$ is $I(\tilde{M})$-invariant which is a closed nowhere-dense subset of $\tilde{M}(\infty)$.

Conversely, if $I(\tilde{M})$ leaves invariant some proper closed subset X of $\tilde{M}(\infty)$. Then by proposition 3.2.6, G^* leaves invariant a proper closed subset of $\tilde{M}(\infty)$

and by theorem 3.2.4, $\tilde{M}$ is isometric to a symmetric space of noncompact type and rank $k \geq 2$. $\square$

Theorem 3.2.8 *Let M be a complete Riemannian manifold with sectional curvature $K \leq 0$ and finite volume, and assume that the universal cover $\tilde{M}$ is irreducible. Then $\tilde{M}$ is isometric to a symmetric space of noncompact type and rank $k \geq 2$ if and only if the geodesic flow in the unit tangent bundle SM does not have a dense orbit.*

Proof : Let $\Gamma \subseteq I(\tilde{M})$ be the lattice such that $M = \tilde{M}/\Gamma$. We use the fact that Γ satisfies the duality condition. Therefore, it follows from proposition 3.1.12 that the geodesic flow in the unit tangent bundle SM has a dense orbit in SM if and only if Γ has a dense orbit in $\tilde{M}(\infty)$. Hence if the geodesic flow has no dense orbit in SM, then $\tilde{M}$ is symmetric of noncompact type and rank $k \geq 2$ by [E3, theorem 4.1].
Conversely, if $\tilde{M}$ is isometric to a symmetric space of noncompace type and rank $k \geq 2$, then $\tilde{M}(\infty)$ admits many proper, closed subsets X that are invariant under $I(\tilde{M})$ and in particular, under Γ. For example, let X be the set of singular points of $\tilde{M}(\infty)$. $\square$

3.3 Rank

3.3.1 Rank of a manifold of nonpositive curvature

Let M be a compact Riemannian manifold of nonpositive sectional curvature. Given $v \in SM$, one defines $r(v)$ to be the dimension of the vector space of parallel Jacobi fields along the geodesic $\gamma_v : \mathbb{R} \longrightarrow M$ which has initial velocity v.

Definition 3.3.1 $rank(M) = min\{r(v) : v \in SM\}$.

The rank of any manifold M is at least 1 since for any $v \in SM$, the velocity field of γ_v is parallel along γ_v. The following proposition contains some basic properties of the rank(See [BBE]).

Proposition 3.3.2 *Let M be any complete manifold of nonpositive sectional curvature, then*

1. $1 \leq rank(M) \leq \dim(M)$.
2. *M is flat if and only if* $rank(M) = \dim(M)$.
3. $rank(M_1 \times M_2) = rank(M_1) + rank(M_2)$, *where $M_1 \times M_2$ denotes the Riemannian product of M_1 and M_2.*

Remark. From (1) and (2), we may regard $rank(M)$ as a measurement of the flatness of M. If M is a symmetric space of noncompact type or a quotient manifold of such a space, then $rank(M)$ coincides with the other definition of rank of M given in section 2.1.8.

In [Ba] and [BS], the following useful result of W. Ballmann and K. Burns-R. Spatzier is proved.

Theorem 3.3.3 *Let $\tilde{M}$ be a complete, irreducible manifold of $K(\tilde{M}) \leq 0$ with $rank(\tilde{M}) \geq 2$ and $K(\tilde{M}) \geq -a^2$ for some positive constant a. If $I(\tilde{M})$ admits a lattice Γ, then $\tilde{M}$ is isometric to a symmetric space of noncompact type and rank $k \geq 2$.*

3.3.2 Rank of the fundamental group

For a complete manifold M of nonpositive sectional curvature, the homotopy information is carried in the fundamental group since the higher homotopy groups vanish. Therefore, it becomes to ask what geometric information of M is contained in the algebraic structure of $\pi_1(M)$.

Let Γ be an abstract group. Given an integer $k \geq 1$, let

$$\begin{aligned} A_k(\Gamma) = \{\phi \in \Gamma \ : \ & Z(\phi) \ contains\ a\ free\ abelian\ subgroup\ of \\ & rank\ r \leq k\ as\ a\ subgroup\ of\ finite\ index\} \end{aligned}$$

where $r \leq k$ is arbitrary and $Z(\phi) = \{\psi \in \Gamma : \psi\phi = \phi\psi\}$.

One defines

$$\begin{aligned} r(\Gamma) = \min\{k \geq 0 \ : \ & \Gamma = \phi_1 A_k(\Gamma) \cup \cdots \cup \phi_m A_k(\Gamma) \\ & where\ \phi_i \in \Gamma, \forall i\} \end{aligned}$$

and set

$$rank(\Gamma) = \sup\{r(\Gamma^*) : \Gamma^* \subseteq \Gamma\ is\ a\ finite\ index\ subgroup\}.$$

In [BE], the following result is proved.

Theorem 3.3.4 *Let M be a complete manifold of finite volume whose sectional curvature satisfies $-a^2 \leq K \leq 0$ for some positive constant a. Then*

$$rank(M) = rank(\pi_1(M)).$$

As an application of theorem 3.3.3 and theorem 3.3.4, one obtains a charaterization of irreducible locally symmetric spaces of noncompact type and rank at least 2 in terms of algebraic data in the fundamental group.

Theorem 3.3.5 *Let M be a complete manifold of finite volume that admits no finite Riemannian cover that splits as a Riemannian product. Assume that the sectional curvature satisfies $-a^2 \leq K \leq 0$ for some positive constant a. Then the following conditions are equivalent:*

1. *The universal Riemannian covering manifold $\tilde{M}$ of M is a symmetric space of noncompact type of rank $k \geq 2$.*

2. *(a) $\pi_1(M)$ is finitely generated.*

   (b) No finite index subgroup of $\pi_1(M)$ is a direct product.

   (c) rank$(\pi_1(M)) = k \geq 2$.

3.4 Rigidity Theorems of Locally Symmetric Spaces

One defines a geometric property of compact connected manifolds of nonpositive sectional curvature to be a rigid property if whenever it holds for a manifold M it also holds for any manifold M^* that is homotopically equivalent to M.

In this section, we will use some facts from the theory of Weyl chamber and Tits geomerty. Then we will give an outline of the proof of the Mostow rigidity theorem and also a proof of its generalization by Gromov.

Definition 3.4.1 *A Riemannian manifold M is called a locally symmetric space if for each $p \in M$, there exists a normal neighborhood of p on which the geodesic symmetry with respect to p is an isometry.*

Definition 3.4.2 *Let X and Y be topological spaces, a continuous function $f : X \longrightarrow Y$ is called homotopy equivalence if there exists a continuous function $g : Y \longrightarrow X$ such that both $f \circ g$ and $g \circ f$ are homotopy to the identity(of the space they are defined on).*

Definition 3.4.3 *Let N be any connceted, complete Riemannian manifold. For any two subsets A and B of N and for each positive number r, one defines the Hausdorff distance*

$$Hd(A, B) = \inf\{r > 0 : A \subseteq T_r(B) \text{ and } B \subseteq T_r(A)\}$$

where

$$T_r(B) = \{x \in N : d(x, B) < r\}$$

is the r-neighborhood of B.

Definition 3.4.4 *A continuous function $f : X \longrightarrow Y$ between metric spaces is called a (k, l)-pseudoisometry(here k, l are positive numbers), if*

1. $d(f(x), f(y)) \leq kd(x, y)$ *for all* $x, y \in X$

2. $d(f(x), f(y)) \geq (1/k)d(x, y)$ *for all* $x, y \in X$ *with* $d(x, y) \geq l$.

If a function f only satisfies 2, then f is called (k, l)-incompressible.

Theorem 3.4.5 (Mostow) *Let M_1 and M_2 be compact locally symmetric spaces of nonpositive curvature. If M_1 and M_2 have isomorphic fundamental group, then up to normalizing constants, M_1 and M_2 are isometric provided that M_1 is irreducible of rank $k \geq 2$.*

Outline of the proof.

For convenience, one divides the proof into several parts.

Let $\tilde{M}_1$ and $\tilde{M}_2$ be the universal coverings of M_1 and M_2, and let Γ_1 and Γ_2 be two suitable discrete subgroups of $G_1 = I_0(\tilde{M}_1)$ and $G_2 = I_0(\tilde{M}_2)$ respectively such that M_1 and M_2 can be regarded as:

$$M_1 = \tilde{M}_1/\Gamma_1 \qquad \textit{and} \qquad M_2 = \tilde{M}_2/\Gamma_2.$$

One also regards the groups Γ_1 and Γ_2 as the fundamental groups of M_1 and M_2.

By hypothesis, there is an isomorphism $\theta : \Gamma_1 \longrightarrow \Gamma_2$.

Part 1.

The proof of Mostow is the construction of a θ-invariant pseudoisometry $f : \tilde{M}_1 \longrightarrow \tilde{M}_2$. The θ-invariant means,

$$f(\gamma_1 x_1) = \theta(\gamma_1) f(x_1) \ for\ all\ x_1 \in \tilde{M}_1\ and\ \gamma_1 \in \Gamma_1.$$

The existence of f follows from the fact, that M_1 and M_2 are homotopy equivalent and the universal covers $\tilde{M}_1$ and $\tilde{M}_2$ are simply-connected.

Part 2.

Let k be the rank of the symmetric space $\tilde{M}_1$. Mostow proves: There is a constant $r > 0$ such that for every flat F_1 in $\tilde{M}_1$, there is a unique flat F_2 in $\tilde{M}_2$ such that $Hd(f(F_1), F_2) \leq r$.

Therefore, the pseudoisometry f induces a map $\bar{f}$ from the flats in $\tilde{M}_1$ to the flats in $\tilde{M}_1$.

Part 3.

One shows that the map $\bar{f}$ from the flats in $\tilde{M}_1$ to the flats in $\tilde{M}_2$ induces an isomorphism of the Tits buildings $\mathcal{T}(\tilde{M}_1)$ and $\mathcal{T}(\tilde{M}_2)$. A theorem of Tits says in the rank $k \geq 2$ case, that the Tits builing determines the symmetric space. Therefore, every isomorphism of the Tits buildings induce an isometry of the symmetric spaces(up to normalizing constant). Hence, $\tilde{M}_1$ and $\tilde{M}_2$ are isometric and the construction implies, that this isometry can be pushed down to an isometry of the quotients M_1 and M_2.

Now, one wants to looking for a general version of this rigidity result, only assuming that M_1 is locally symmetric and M_2 is allowed to be an arbitrary manifold of nonpositive curvature.

Here, one cannot allow M_1 to be a locally symmetric space of rank one because a rank one symmetric space has strictly negative curvature and there are nearby nonsymmetric metrics on M_1. But in the higher rank case, the existence of the flat subspaces makes the rigidity plausible.

Theorem 3.4.6 (Gromov) *Let $\tilde{M}^*$ be an irreducible symmetric space of rank $k \geq 2$, and let $M^* = \tilde{M}^*/\Gamma^*$ be a compact locally symmetric space of nonpositive curvature. Let $M = \tilde{M}/\Gamma$ be a compact manifold of nonpositive curvature whose fundamental group is isomorphic to the fundamental group of M^*. Then M^* and M are isometric if the metric of M^* is rescaled by a suitable constant.*

Proof.

Step 1:

By hypothesis, there exists an isomorphism $\theta : \Gamma^* \longrightarrow \Gamma$. Therefore M^* and M are homotopy equivalent. Then let $\bar{f} : M^* \longrightarrow M$ and $\bar{g} : \tilde{M} \longrightarrow \tilde{M}^*$ be maps such that $\bar{g} \circ \bar{f}$ and $\bar{f} \circ \bar{g}$ are homotopic to the identities on $\tilde{M}^*$ and M.

Let $f : \tilde{M}^* \longrightarrow \tilde{M}$ and $g : \tilde{M} \longrightarrow \tilde{M}^*$ be the lifts to the covering spaces. Then there are constants $l, m > 0$ such that f and g are (l, m)-pseudoisometries. Clearly, $f(\gamma^* x^*) = \theta(\gamma^*) f(x^*)$ for $x^* \in \tilde{M}^*$ and $\gamma^* \in \Gamma^*$ and $g(\gamma x) = \theta^{-1}(\gamma) g(x)$ for $x \in \tilde{M}$ and $\gamma \in \Gamma$.

Furthermore, there is a constant $C > 0$ such that $d(x, gf(x^*)) \leq C$ and $d(x, fg(x)) \leq C$ for $x^* \in \tilde{M}^*$ and $x \in \tilde{M}$.

Lemma 3.4.7 *Let $f : X \longrightarrow Y$ be an incompressible map between two simply connected complete manifolds of nonpositive curvature, then* $\dim Y \geq \dim X$ *and if the dimensions are equal, then f is surjective.*

This result implies, that the manifolds $\tilde{M}^*$ and $\tilde{M}$ have the same dimensions and that the maps f and g are surjective.

Step 2:

Let $k \geq 2$ be the rank of the symmetric space $\tilde{M}^*$, then a flat in $\tilde{M}^*$ is a totally geodesic embedded flat subspace of dimension k. The following result is due to Mostow.

Lemma 3.4.8 *There is a constant $r \in \mathbb{R}$ such that for any flat F^* in $\tilde{M}^*$, there is a k-flat F in $\tilde{M}$ with $Hd(f(F^*), F) \leq r$.*

Remark. One calls a k-flat F with $Hd(f(F^*), F) \leq r$ an image-flat of F^*. In Mostow's case, the image-flat is unique. In contrast to that situation, it is not yet clear, that an image-flat of F^* is uniquely determined. If F_1 and F_2 are images of F^*, then $Hd(F_1, F_2) \leq 2r$ and one shows that F_1 is parallel to F_2, and F_1, F_2 bound a family of flats all with finite Hausdorff distance to $f(F^*)$. On the other hand, the preimage of an image-flat is unique: In the symmetric space $\tilde{M}^*$ one has $Hd(F_1^*, F_2^*) = \infty$ for images F_i of F_i^*. Thus every image-flat F in $\tilde{M}$ determines a unique preimage F^* in $\tilde{M}^*$.

Lemma 3.4.9 *The set of image-flat is closed in the set of all k-flats of $\tilde{M}$. i.e. let F_i be a sequence of image flats $F_i \to F, F$ a k-flat in $\tilde{M}$, let F_i^* be the preimages of F_i, then $F_i^* \to F^*$ and $Hd(f(F^*), F) \leq r$.*

Proof : Pick $x \in F$, then there are $x_i \in F_i$ with $d(x_i, x) \to 0$. We can find points $x^* \in F_i^*$ with $d(f(x_i^*), x_i) \leq r$, hence $d(f(x_i^*), x) \leq r + 1$ for i large. Then

$$d(x_i^*, g(x)) \leq d(gf(x_i^*), g(x)) + a \leq l(r+1) + a$$

for some $a > 0$. Thus the sequence $x_i^* \in F_i^*$ is bounded and hence there is a subsequence F_{ij}^* converging to a k-flat F^*.
One proves, that $Hd(f(F^*), F) \leq r$:

(i) For $y^* \in F^*$ there are $y_i^* \in F_{ij}^*, y_j^* \to y^*$. Then $d(f(y_{ij}^*), F_{ij}) \leq r$ and by continuity $d(f(y^*), F) \leq r$.

(ii) If $y \in F$, then there are $y_j \in F_{ij}$ with $y_j \to y$. There are $y_j^* \in F_{ij}^*$ with $d(f(y_j^*), y_j) \leq r$. Because

$$\begin{aligned} d(y_j^*, g(y)) &\leq d(gf(y_j^*), g(y)) + a \\ &\leq ld(f(y_j^*), y) + a \\ &\leq l(d(y_j, y) + r) + a, \end{aligned}$$

the sequence y_j^* is bounded and hence has an accumulation point $y^* \in F^*$, then $d(f(y^*), y) \leq r$.

By the above remark, F^* is uniquely determined by F. Hence the above argument shows, that every subsequence of F_i^* has a subsequence converging to F^*, thus $F_i^* \to F^*$. □

Lemma 3.4.10 *Let F^* be a flat in $\tilde{M}^*$, F an image-flat of F^*, $\bar{f} : \pi_F \circ f : F^* \longrightarrow F$, where π_F is the projection onto the k-flat F. Then $\bar{f}$ is a $(2l, \max(4rl, m))$-pseudoisometry.*

Proof : (i) $d(\bar{f}(x^*), \bar{f}(y^*)) \leq d(f(x^*), f(y^*)) \leq ld(x^*, y^*)$ because π_F is distance decreasing.

(ii) Since $d(f(x^*), \bar{f}(x^*)) \leq r$, one has

$$d(\bar{f}(x^*), \bar{f}(y^*)) \geq d(f(x^*), f(y^*)) - 2r.$$

Then for $x^*, y^* \in F^*$ with $d(x^*, y^*) \geq \max(4rl, m)$, one obtains

$$\begin{aligned} (1/2l)d(x^*, y^*) &\leq 1/l(d(x^*, y^*) - 2rl) \\ &\leq d(f(x^*), f(y^*)) - 2r \\ &\leq d(\bar{f}(x^*), \bar{f}(y^*)). \end{aligned}$$ □

Step 3:

<u>3.1</u> Given a point $x \in \tilde{M}$, one constructs an involution $\psi_x : \mathcal{T}(\tilde{M}^*) \longrightarrow \mathcal{T}(\tilde{M}^*)$ of the Tits building of $\tilde{M}^*$. By Tits' theorem, ψ_x is induced by an isometry $\Phi_x : \tilde{M}^* \longrightarrow \tilde{M}^*$. One proves that Φ_x is the geodesic symmetry at a point $x^* \in \tilde{M}^*$. Thus one can define a map

$$\begin{aligned} \Phi : M &\longrightarrow \tilde{M}^* \\ x &\mapsto x^* \end{aligned}$$

Before one defines the involution ψ_x of the Tits building one need some useful facts.

Lemma 3.4.11 *Let $w \subset \tilde{M}^*(\infty)$ be a Weyl chamber at infinity, and let c^* be a geodesic in $\tilde{M}^*$ with $c^*(\infty) \in w$. Let $p \in \tilde{M}$ and c a geodesic parametrized by arc length with $c(0) = p$ and $c'(0)$ is an accumulation point of the initial vectors of geodesics from p to $f(c^*(t_i))$ as $t_i \to \infty$. Then there is a unique flat F^* in $\tilde{M}^*$ with:*

1. *$w \subset F^*(\infty)$.*
2. *the function $t \mapsto d(c(t), F)$ is constant, where F is any image-flat of F^*.*

Lemma 3.4.12 *Let $x \in \tilde{M}, w \subset \tilde{M}^*(\infty)$ a Weyl chamber at infinity. Then there is a unique flat F^* in $\tilde{M}^*$ with:*

1. *$w \subset F^*(\infty)$*

2. there is an image-flat F of F^* with $x \in F$. Furthermore, also the image-flat F with $x \in F$ is uniquely determined.

3.2 Now, one can define the involution ψ_x. Let $x \in \tilde{M}$ be a given point. let b be an s-simplex of the Tits building $\mathcal{T}(\tilde{M}^*)$, then there is a Weyl chamber w with $b \leq w$(i.e. b is a wall of w). By lemma 3.4.12, there is a unique flat F^* in $\tilde{M}^*$ with $w \subset F^*(\infty)$ and a unique k-flat F in $\tilde{M}$ with $Hd(f(F^*), F) \leq r$ and $x \in F$.

One defines $\psi_x(b)$ to be the antipodal Weyl chamber(wall) in the flat F^*. Thus, if $c_1^*, \dots, c_s^*$ are maximal singular geodesic in F^*, such that $c_1^*(\infty), \dots, c_s^*(\infty)$ span b, then $\psi_x(b)$ is the simplex spanned by $c_1^*(-\infty), \dots, c_s^*(-\infty)$.

Finally, one proves that the map ψ_x is well defined.

Lemma 3.4.13 *Let $b \in \mathcal{T}(\tilde{M}^*)$ be a 1-simplex, $b \leq w_1, b \leq w_2$. Let F_i^* be the unique flats in $\tilde{M}^*$ with $w_i \subset F_i^*(\infty)$ such that there is an image F_i with $x \in F_i, i = 1, 2$. Let c^* be a maximal singular geodesic in F_1^* with $c^*(\infty) = b$. Then there is a maximal singular geodesic $\tilde{c}^*$ in F_2^* parallel to c^*.*

3.3 By definition, the map ψ_x respects the order structure of $\mathcal{T}(\tilde{M}^*)$ and ${\psi_x}^2 = id$. Thus one can use the theorem of Tits that ψ_x is induced by an isometry $\Phi_x : \tilde{M}^* \longrightarrow \tilde{M}^*$. Since ${\psi_x}^2 = id$, Φ_x has order 2. Hence Φ_x has a fixed point $x^* \in \tilde{M}^*$. One also proves that x^* is the unique fixed point. Therefore, one can define the map

$$\begin{aligned} \Phi : \tilde{M} &\longrightarrow \tilde{M}^* \\ x &\mapsto x^* \end{aligned}$$

where x^* is the unique fixed point of Φ_x.

For a detailed reference, see [Mo, p.120, p.123 and p.126].

Step 4:

4.1 If F^* is a flat such that there is an image flat F with $x \in F$, then by definition of ψ_x, the Weyl chambers in $F^*(\infty)$ are only permuted by ψ_x. Thus Φ_x leaves F^* invariant and Φ_x is a reflection of F^*. Hence for $x^* \in F^*$, one can see that Φ_x is the geodesic symmetry at x^*.
For $x \in \tilde{M}$, let

$$\mathcal{F}_x = \{ \text{ image flats } F \text{ in } \tilde{M}, x \in F \}$$

then one can show the following results.

Lemma 3.4.14 *Let $\mathcal{F}_x^*$ be the set of preimages of the flats in $\mathcal{F}_x$. Then $x^* = \Phi(x)$ is the unique point with $x^* \in F^*$ for all $F^* \in \mathcal{F}_x^*$.*

Lemma 3.4.15 *$\Phi : \tilde{M} \longrightarrow \tilde{M}^*$ is a homemorphism which satisfies $\Phi(\gamma x) = \theta^{-1}(\gamma)\Phi(x)$ for all $\gamma \in \Gamma$.*

Now, let F^* be a flat in $\tilde{M}^*$ and F be an image-flat of F^*. Then $Hd(f(F^*), F) \leq r$. If $x \in F$, then $\Phi(x) \in F^*$. Therefore $\Phi|_F : F \longrightarrow F^*$ is an incompressible map and thus surjective. Also, $\Phi|_F$ is a homeomorphism of F onto F^*. Let

$$\Psi = \Phi^{-1} : \tilde{M}^* \longrightarrow \tilde{M},$$

if F^* is a flat in $\tilde{M}^*$, then $\Psi|_{F^*}$ is a homeomorphism of F^* onto a k-flat $\Psi(F^*)$ with $x = \Psi(x^*) \in \Psi(F^*) = F$.
Let E^* be a singular linear subspace in $F^*, x^* \in E^*$. Then $E^* = F_1^* \cap \cdots \cap F_s^*$ for some flats F_i^* with $x^* \in F_i^*$ and $\Psi|_{E^*}$ is a homeomorphism if E^* onto $E = \Psi(F_1^*) \cap \cdots \cap \Psi(F_s^*)$. Thus E is an intersection of k-flats and is a linear

subspace of F. In particular, a maximal singular geodesic c^* in F^* is mapped homeomorphically onto a geodesic $c = \Psi(c^*)$ in F.
Now, let $c_1^*, \ldots, c_k^*$ be maximal singular geodesics, parametrized by arc length. Then $c_i^*(0) = x^*$ and $c_i^{*\prime}(0)$ form a basis of $T_{x^*}F^*$. Then any point $y^* \in F^*$ can be uniquely expressed as

$$y^* = c_1^*(s_1) + \cdots + c_k^*(s_k) \qquad \text{for } s_i \in \mathbb{R}$$

where one identifies $F^* \simeq \mathbb{R}^k, x^* \simeq 0$.
Furthermore, with this identification $\Psi|_{F^*} : F^* \longrightarrow F$ is a linear map.

4.2 The pointed flat (F^*, x^*) is divided into the Weyl chambers by the singular hyperplanes through x^*. Since the map $\Psi|_{F^*} : (F^*, x^*) \longrightarrow (F, x)$ is an invertible linear map, the image-flat F is divided by the images of the singular hyperplanes into cones, which one calls Weyl chambers in (F, x). Therefore, one has an induced Weyl chamber structure in the pointed flat (F, x) and Ψ induces a map on the set of Weyl chambers.

Lemma 3.4.16 *Let (F^*, x^*) and (F, x) be pointed flats, then:*

1. *All image Weyl chambers in a point flat (F, x) are isometric.*

2. *There is an orthogonal map $S : (F, x) \longrightarrow (F^*, x^*)$ such that $S(\Psi(H^*)) = H^*$ for all singular hyperplane in (F^*, x^*) and $S(\Psi(w)) = w$ for all Weyl chamber w in (F^*, x^*).*

3. *If the metric on F^* is scaled by a positive constant λ, then $\Psi : (F^*, x^*) \longrightarrow (F, x)$ is an isometry, i.e. $d(\Psi(x), \Psi(y)) = \lambda d(x, y)$ for all $x, y \in F^*$.*

Now choose $x^* \in \tilde{M}^*$, let (F^*, x^*) and $(\tilde{F}^*, x^*)$ be point flats such that $F^* \cap \tilde{F}^*$ contains a geodesic. Then the constants λ and $\tilde{\lambda}$ of the above lemma are equal.

For any two flats $F^*, \tilde{F}^*$ with $x^* \in F^* \cap \tilde{F}^*$, there is a sequence $F_1^*, \ldots, F_s^*$ with $F_1^* = F^*$ and $F_s^* = \tilde{F}^*$ such that $F_i^* \cap F_{i+1}^*$ is a singular hyperplane. Hence the constant λ is the same for all flats F^* with $x^* \in F^*$.

Since $y^* \in \tilde{M}^*$, there is a flat F^* with $x^*, y^* \in F^*$, the constant is also the same for all flats in $\tilde{M}^*$. Thus the above lemma implies that $d(\Psi(x^*), \Psi(y^*)) = \lambda d(x^*, y^*)$ for all points $x^*, y^* \in \tilde{M}^*$. Thus, if one scales the metric, then Ψ is a θ-invariant isometry from $\tilde{M}^*$ to $\tilde{M}$ and therefore induces an isometry from $\tilde{M}^*$ to $\tilde{M}$.

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